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Solitary Wave Solutions to Approximate Fully Nonlinear Double sine-Gordon Equation

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Abstract: By using ansatz method to solve the approximate fully nonlinear double sine-Gordon equation, some exact traveling wave solutions (compacton, peakon) can be derived. Especially a type of discontinuous solution to approximate fully nonlinear double sine-Gordon equation (AFNDSG) is obtained. The noncontinuous solitary wave solutions are verified by applying conservation law theory.

Key words:double sine-Gordon equation; noncontinuous solitary wave solution; conservation law

1 Introduction

The study of solitons is an important part in the solitary wave theory. Recently, two kinds of special solitons, compacton and peakon, were found. In order to study the role of nonlinear dispersive term in the formation of patterns in liquid drops, Rosenau and Hyman [1] studied the generalized nonlinear dispersive equation $K(m, n)$ given by

$$u_t + (u^m)_x + (u^n)_{xxx} = 0$$

They obtained solitary wave solutions with compact support in it which they called compacton. Other studies have been shown in [2-10]. Camassa and Holm [11] found peakon solutions, which are called peakon because they have discontinuous first-order derivative at the wave peaks. More exact solutions of double sine-Gordon equation have been obtained by using F-expansion method [12]. The Jacobi-sn and Jacobi-cn function solutions to double sine-Gordon equation were obtained by Fan and Zhang [13] who used the Jacobi elliptic function expansion with symbolic computation. In this paper, Many exact solutions, especially a type of noncontinuous solitary wave solution, are obtained. In order to study the role of nonlinear term, we introduce the conception of nonlinear intensity to investigate the approximate fully nonlinear double sine-Gordon equation (AFNDSG)

$$(u^m)_{tt} - (u^n)_{xx} + \alpha u^p - \frac{\alpha u^{3p}}{3!} + \beta(2u)^p - \frac{\beta(2u)^{3p}}{3!} = 0 \tag{1}$$

compacton, noncontinuous solitary wave solutions and peakon solutions will be found, which are of great significance to show the role of nonlinear term and the varieties of solutions. Then we prove that verify these solutions are noncontinuous solitary wave solutions.

The rest of this paper is organized as follows: In Section 2, the conservation law of AFNDSG equation is given and compacton solutions, noncontinuous solitary wave solutions and peakon solutions of AFNDSG are obtained by using ansatzs method. In Section 3, a conclusion is given.

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2 New types of solitary wave solutions and conservation law of AFNDSG equation

Approximate double sine-Gordon equation (ADSG) was studied in [14-15]. Feng [14] gave a qualitative analysis to an approximate sine-Gordon equation using the qualitative theory of ODE. Exact solutions to the approximate $(n+1)$-dimensional sine-Gordon equation were expressed. Wang [15] constructed solitary wave solutions to the approximate equation (2) by using a homogeneous balance method.

$$u_{tt}-u_{xx}+(\alpha+2\beta)u-\frac{\alpha u^3}{3!}-\frac{4\beta u^3}{3}=0 \tag{2}$$

In order to investigate the role of nonlinear terms, we introduce the approximate fully nonlinear double sine-Gordon equation (AFNDSG)

$$(u^m)_{tt}-(u^n)_{xx}+\alpha u^p-\frac{\alpha u^{3p}}{3!}+\beta(2u)^p-\frac{\beta(2u)^{3p}}{3!}=0$$

where α,β are arbitrary constants, and m,n,p are nonlinear intensity of Eq.(1). When $m=n=p=1,\quad a=1$, Eq. (1) equals Eq. (2). Set $\xi=x-ct$, and Eq. (1) becomes

$$c^2(u^m)_{\xi\xi}-(u^n)_{\xi\xi}+\alpha u^p+\beta(2u)^p-\left(\frac{\alpha u^{3p}}{3!}+\frac{\beta(2u)^{3p}}{3!}\right)=0 \tag{3}$$

2.1 The conservation law of AFNDSG equation

First, we give the conservation law of AFNDSG equation.

Let $u_x=v$, $u_t=w$, thus $v_t=w_x=u_{xt}$. So, AFNDSG equation can be transformed into the following equation sets

$$\begin{cases} w_t=\frac{nu^{n-1}}{mu^{m-1}}v_x-\frac{m-1}{u}w^2+\frac{n(n-1)u^{n-m-1}}{m}v^2-\frac{\alpha u^{p-m+1}}{m}+\frac{\alpha u^{3p-m+1}}{3!m}-\frac{\beta(2u)^p}{mu^{m-1}}+\frac{\beta(2u)^{3p}}{3mu^{m-1}}, \\ v_t=w_x, \\ u_t=w. \end{cases} \tag{4}$$

When $m=n$, (4) equals to

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}_t=\begin{bmatrix} w \\ w_x \\ v_x-\frac{n-1}{u}w^2+\frac{(n-1)}{u}v^2-\frac{\alpha u^{p-n+1}}{n}+\frac{\alpha u^{3p-n+1}}{3!n}-\frac{\beta(2u)^p}{nu^{n-1}}+\frac{\beta(2u)^{3p}}{3nu^{n-1}} \end{bmatrix}.$$

The relevant conservation law can be rewritten to be

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}_t-\begin{bmatrix} 0 \\ w \\ v \end{bmatrix}_x=\begin{bmatrix} w \\ 0 \\ -\frac{n-1}{u}w^2+\frac{(n-1)}{u}v^2-\frac{\alpha u^{p-n+1}}{n}+\frac{\alpha u^{3p-n+1}}{3!n}-\frac{\beta(2u)^p}{nu^{n-1}}+\frac{\beta(2u)^{3p}}{3nu^{n-1}} \end{bmatrix}.$$

2.2 Noncontinuous solution and compacton solution to AFNDSG equation

Now, we consider the AFNDSG equation (3) and find its solitary wave solutions using the ansatzs method.

Case 1. When $m=n\neq p$,

$$(c^2-1)(u^m)_{\xi\xi}+\alpha u^p+\beta(2u)^p-\frac{\alpha u^{3p}}{3!}-\frac{\beta(2u)^{3p}}{3!}=0. \tag{5}$$

We set the form of solution as follows

$$u=\begin{cases} A\cos^\delta B\xi, & |B\xi|\le\frac{\pi}{2}, \\ 0, & |B\xi|>\frac{\pi}{2}. \end{cases} \tag{6}$$

Substitute (6) into Eq.(5), and this gives

$$\begin{aligned}&(c^2-1)m\delta(m\delta-1)A^mB^2\cos^{m\delta-2}B\xi-(c^2-1)m^2\delta^2A^mB^2\cos^{m\delta}B\xi+(\alpha+2^p\beta)A^p\cos^{p\delta}B\xi\\&-\left(\tfrac{\alpha+8^p\beta}{3!}\right)A^{3p}\cos^{3p\delta}B\delta=0.\end{aligned}\tag{7}$$

Collecting the coefficients of each power of trigonometric function and setting them to zero, we obtain

$$\left\{\begin{array}{l}m\delta-2=p\delta,\\ m\delta=3p\delta,\\ (c^2-1)m\delta(m\delta-1)A^mB^2=(\alpha+2^p\beta)A^p,\\ (c^2-1)m^2\delta^2A^mB^2=\frac{(\alpha+\beta8^p)}{3!}A^{3p}.\end{array}\right.\tag{8}$$

Solving (8) with Maple, we have

$$m=3p,\quad \delta=\frac{1}{p},\quad B^2=\frac{(\alpha+\beta8^p)}{54(c^2-1)},\quad A^{2p}=\frac{9(\alpha+2^p\beta)}{(\alpha+8^p\beta)}\tag{9}$$

Substituting (9) into (7), we then obtain the compacton solution, for $m=n\neq p$

$$u=\left\{\begin{array}{ll}\pm3\sqrt{\frac{\alpha+2^p\beta}{\alpha+8^p\beta}}\left\{\cos\left(\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right)\right\}^{\frac{1}{p}}, & \left|\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right|\leq\frac{\pi}{2},\\ 0, & \left|\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right|>\frac{\pi}{2}.\end{array}\right.$$

where $\xi=x-ct$, as is shown in Fig.(1) with $p=1$, $\alpha=2$, $\beta=1$, $c=2$.

On the other hand, if we set the form of solution as follows

$$u=\left\{\begin{array}{ll}A\sin^\delta B\xi, & |B\xi|\leq\pi,\\ 0, & |B\xi|>\pi.\end{array}\right.\tag{10}$$

Substituting (10) into Eq. (7), we can also obtain the compacton solution

$$u=\left\{\begin{array}{ll}\pm3\sqrt{\frac{\alpha+2^p\beta}{\alpha+8^p\beta}}\left\{\sin\left(\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right)\right\}^{\frac{1}{p}}, & \left|\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right|\leq\pi,\\ 0, & \left|\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right|>\pi.\end{array}\right.$$

Remark We can obtain a new type of noncontinuous solution

$$u=\left\{\begin{array}{ll}\pm3\sqrt{\frac{\alpha+2^p\beta}{\alpha+8^p\beta}}\left\{sgn(\xi)\cos\left(\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right)\right\}^{\frac{1}{p}}, & \left|\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right|\leq\frac{\pi}{2},\\ 0, & \left|\sqrt{\frac{(\alpha+\beta8^p)}{54(c^2-1)}}\xi\right|>\frac{\pi}{2}.\end{array}\right.\tag{11}$$

where $\xi=x-ct$, as is shown in Fig.(2) with $p=1,\alpha=2,\beta=1,c=2$.

Case 2. When $m=p\neq n$,

$$c^2(u^m)_{\xi\xi}-(u^n)_{\xi\xi}+\alpha u^m+\beta(2u)^m-\frac{\alpha u^{3m}}{3!}-\frac{\beta(2u)^{3m}}{3!}=0\tag{12}$$

According to similar analysis to the above, we set $n=3m$, and the forms of solutions can be shown as follows

$$\left\{\begin{array}{l}u^m=A\cos B\xi\\ u^n=A^3\cos^3B\xi\end{array}\right.\tag{13}$$

$$\left\{\begin{array}{l}u^m=A\sin B\xi\\ u^n=A^3\sin^3B\xi\end{array}\right.\tag{14}$$

Substituting (13) and (14) into Eq.(12) and using the same method, we obtain the compacton solutions

$$u=\begin{cases} \pm\sqrt{\frac{9(\alpha+2^m\beta)}{\alpha+8^m\beta}-\frac{c^2}{6}}\left\{\cos\left(\sqrt{\frac{(\alpha+\beta 8^m)}{54}}\xi\right)\right\}^{\frac{1}{m}}, & \left|\sqrt{\frac{(\alpha+\beta 8^m)}{54}}\xi\right|\le\frac{\pi}{2},\\ 0, & \left|\sqrt{\frac{(\alpha+\beta 8^m)}{54}}\xi\right|>\frac{\pi}{2}.\end{cases}$$

$$u=\begin{cases} \pm\sqrt{\frac{9(\alpha+2^m\beta)}{\alpha+8^m\beta}-\frac{c^2}{6}}\left\{\sin\left(\sqrt{\frac{(\alpha+\beta 8^m)}{54}}\xi\right)\right\}^{\frac{1}{m}}, & \left|\sqrt{\frac{(\alpha+\beta 8^m)}{54}}\xi\right|\le\pi,\\ 0, & \left|\sqrt{\frac{(\alpha+\beta 8^m)}{54}}\xi\right|>\pi.\end{cases}$$

Case 3. When $m\neq n=p$

$$c^2(u^m)_{\xi\xi}-(u^n)_{\xi\xi}+\alpha u^n+\beta(2u)^n-\frac{\alpha u^{3n}}{3!}-\frac{\beta(2u)^{3n}}{3!}=0$$

Set $m=3n$ in (12), the forms of solutions are as follows

$$\begin{cases} u^n=A\cos B\xi\\ u^m=A^3\cos^3 B\xi\end{cases}$$

$$\begin{cases} u^n=A\sin B\xi\\ u^m=A^3\sin^3 B\xi\end{cases}$$

Using the same method, we can also obtain compacton solutions

$$u=\begin{cases} \pm\sqrt{\frac{-9(\alpha+2^n\beta)}{\alpha+8^n\beta}-\frac{1}{6}}\left\{\cos\left(\sqrt{\frac{(\alpha+\beta 8^n)}{54}}\xi\right)\right\}^{\frac{1}{n}}, & \left|\sqrt{\frac{(\alpha+\beta 8^n)}{54}}\xi\right|\le\frac{\pi}{2},\\ 0, & \left|\sqrt{\frac{(\alpha+\beta 8^n)}{54}}\xi\right|>\frac{\pi}{2}.\end{cases}$$

$$u=\begin{cases} \pm\sqrt{\frac{-9(\alpha+2^n\beta)}{\alpha+8^n\beta}-\frac{1}{6}}\left\{\sin\left(\sqrt{\frac{(\alpha+\beta 8^n)}{54}}\xi\right)\right\}^{\frac{1}{n}}, & \left|\sqrt{\frac{(\alpha+\beta 8^n)}{54}}\xi\right|\le\pi,\\ 0, & \left|\sqrt{\frac{(\alpha+\beta 8^n)}{54}}\xi\right|>\pi.\end{cases}$$

Figure 1: Compacton solution

2.3 The verification of noncontinuous solutions of AFNDSG equation

Theorem The solution(11) of AFNDSG equation

$$u=\begin{cases} \pm 3\sqrt{\frac{\alpha+2^p\beta}{\alpha+8^p\beta}}\left\{sgn(\xi)\cos\left(\sqrt{\frac{(\alpha+\beta 8^p)}{54(c^2-1)}}\xi\right)\right\}^{\frac{1}{p}}, & \left|\sqrt{\frac{(\alpha+\beta 8^p)}{54(c^2-1)}}\xi\right|\le\frac{\pi}{2},\\ 0, & \left|\sqrt{\frac{(\alpha+\beta 8^p)}{54(c^2-1)}}\xi\right|>\frac{\pi}{2}.\end{cases}$$

Figure 2: Noncontinuous solution

is a noncontinuous solution (where $\xi = x - ct$).

Proof(1) It is easily known that (11) is the classical solution of the AFNDSG equation in the smooth parts.

(2) It is easily known that the discontinuous line is $\xi = 0$ when $p = 1$.

We will prove that the expression $[u^*]s = [f]$ holds on discontinuous lines, where $u^* \equiv \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $f = \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}$.

$(i) \quad [u] = u(0^+, t) - u(0^-, t) = 0.$

$$(ii) \quad v = u_x = u_\xi = \pm\sqrt{\frac{\alpha + 2^p\beta}{\alpha + 8^p\beta}} \left\{ \frac{1}{p} sgn(\xi)^{\frac{1}{p}-1} 2\delta \cos^{\frac{1}{p}} \left(\sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \xi \right) + \frac{1}{p} sgn\left(\xi\right)^{\frac{1}{p}} \cos^{\frac{1}{p}-1} \left(\sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \xi \right) \left[- \sin \left(\sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \xi \right) \right] \sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \right\}$$

$\therefore [v] = 0.$

$$(iii) \quad w = u_t = -cu_\xi = \pm 3c\sqrt{\frac{\alpha+2^p\beta}{\alpha+8^p\beta}} \left\{ \frac{1}{p} sgn(\xi)^{\frac{1}{p}-1} 2\delta \cos^{\frac{1}{p}} \left(\sqrt{\frac{(\alpha+8^p\beta)}{54(c^2-1)}} \xi \right) + \frac{1}{p} sgn\left(\xi\right)^{\frac{1}{p}} \cos^{\frac{1}{p}-1} \left(\sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \xi \right) \left[- \sin \left(\sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \xi \right) \right] \sqrt{\frac{(\alpha + 8^p\beta)}{54\left(c^2 - 1\right)}} \right\}$$

$\therefore [w] = 0$

To summarize, $[u^*] = 0, \quad [f] = 0$, namely, (2) holds. This completes the proof of the theorem.

2.4 Peakon solution and noncontinuous solution to AFNDSG equation

Set $u = \lambda e^{-b|\xi|} (b > 0)$, and $\xi = x - ct$. Substituting it into Eq. (3), this gives

$$m^2b^2c^2\lambda^m e^{-mb|\xi|} - \lambda^n e^{-nb|\xi|} b^2 n^2 + \alpha\lambda^p e^{-bp|\xi|} + 2^p\beta e^{-bp|\xi|} - \frac{\alpha\lambda^{3p}e^{-3pb|\xi|}}{6} - \frac{2^{3p}\beta\lambda^{3p}e^{-3pb|\xi|}}{6} = 0$$

Case 1 When $m = 3p$, $p = n$, collecting the coefficients of the same terms, we obtain

$$\begin{cases} m^2b^2c^2\lambda^m - \frac{\alpha\lambda^{3p}}{6} - \frac{2^{3p}\beta\lambda^{3p}}{6} = 0 \\ -\lambda^n b^2 n^2 + \alpha\lambda^p + 2^p\beta = 0 \end{cases}$$

$$b^2 = \frac{\alpha\lambda^n + 2^n\beta}{\lambda^n n^2}, \quad c^2 = \frac{(\alpha + 8^n\beta)\lambda^n}{54(\alpha\lambda^n + 2^n\beta)}$$

Then the peakon solution to Eq.(3) is as follows

$$u = \lambda e^{-\sqrt{\frac{\alpha\lambda^n+2^n\beta}{\lambda^n n^2}}\,x-\sqrt{\frac{(\alpha+8^n\beta)\lambda^n}{54(\alpha\lambda^n+2^n\beta)}}\,t}$$

where λ is arbitrary constant, which is shown in Fig.(3) with $\lambda = 1, \alpha = 2, \beta = 1, n = 1$.

Figure 3: Peakon solution

Case 2 When $n = 3m, p = m$, we can also obtain the peakon solution

$$u = \lambda e^{-\sqrt{\frac{-(\alpha\lambda^n+8^m\beta)}{54m^2}}\,x-\sqrt{54\frac{(\alpha\lambda^m+2^m\beta)}{(\alpha+8^m\beta)\lambda^m}}\,t}$$

With the development of solitary solution theory, many new types of solutions have been found. We find that there is also noncontinuous solution to the AFNDSG equation. We set the solution in the following form:

$$u = \lambda sgn(\xi)e^{-b|\xi|} = \begin{cases} -\lambda e^{b\xi}, & \xi < 0, \\ 0, & \xi = 0, \\ \lambda e^{-b\xi}, & \xi > 0. \end{cases} \tag{15}$$

where $\xi = x - ct$, c is the wave speed and λ is the arbitrary constant. Substituting (15) into Eq. (3), we can obtain the noncontinuous solution by using similar analysis to the above

$$u = \lambda sgn(\xi)e^{-b|\xi|} = \begin{cases} -\lambda e^{\sqrt{\frac{-(\alpha\lambda^n+8^m\beta)}{54m^2}}\,x-\sqrt{54\frac{(\alpha\lambda^m+2^m\beta)}{(\alpha+8^m\beta)\lambda^m}}\,t}, & \xi < 0, \\ 0, & \xi = 0, \\ \lambda e^{-\sqrt{\frac{-(\alpha\lambda^n+8^m\beta)}{54m^2}}\,x-\sqrt{54\frac{(\alpha\lambda^m+2^m\beta)}{(\alpha+8^m\beta)\lambda^m}}\,t}, & \xi > 0. \end{cases}$$

where $\xi = x - \sqrt{54\frac{(\alpha+2^m\beta)}{(\alpha+8^m\beta)}}t$, which is shown in Fig.(4)

Figure 4: Noncontinuous solution

3 Conclusion

By using ansatz method to solve the approximate fully double sine-Gordon equation, some exact traveling wave solutions, especially a type of discontinuous solitary wave solution, are derived. The noncontinuous solitary wave solutions are verified by using conservation law theory.

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