

Flat Chain and Flat Cochain Related to Koch Curve

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Abstract: The theory of flat chain and flat cochain is discussed to solve a problem that how to get the integration with respect to a Lipschitz form defined on the Koch curve.

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It is well known that the integral of any differential form along any smooth curve can be defined. But how to define the integral of some Lipschitz form along some fractal curve, for example, the Koch curve? Please see [1-6] for fractal and chaos.

This paper defines the integral of any Lipschitz form along the Koch curve. Here we say ω is a Lipschitz form, if $\omega = f dx + g dy$ with f, g being Lipschitzian.

1 Flat Chain

Simplex with its boundary [7]

In Euclidean space \mathbb{R}^n , we consider the simplex of dimension r with $r \leq n$ (r -simplex in short), which is a convex polyhedron consisting of $r + 1$ vertexes. As in Figure 1, we have 0-simplex, 1-simplex and 2-simplex...

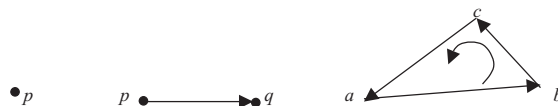


Figure 1: 0-simplex, 1-simplex and 2-simplex

It is noticed that 2-simplex is oriented. Similarly, we think r -simplex $v_0 v_1 \cdots v_r$ and $v_{i_0} v_{i_1} \cdots v_{i_r}$ are the same if the permutation $i_0 i_1 \cdots i_r$ on $\{0, \cdots, r\}$ is even, and $v_0 v_1 \cdots v_r = -v_{i_0} v_{i_1} \cdots v_{i_r}$ if permutation $i_0 i_1 \cdots i_r$ is odd.

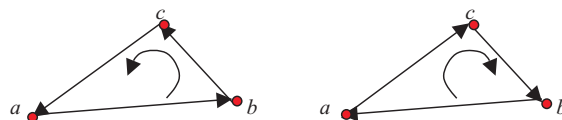


Figure 2: orientation of 2-simplex, $abc = -acb$

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The boundary of simplex $v_0v_1 \cdots v_r$ is defined by

$$\partial(v_0v_1 \cdots v_r) = \sum_i (-1)^i v_0 \cdots v_{i-1}v_{i+1} \cdots v_r,$$

that means the boundary of r -simplex consists of linear combination of $(r - 1)$ -simplexes.

Polyhedron with its boundary [7]

Spanning all the r -dimensional simplexes to be a linear space on \mathbb{R} , we call any element A of this space a r -dimensional Polyhedron, and we can write

$$A = a_1\sigma_1 + \cdots + a_n\sigma_n \text{ with } a_i \in \mathbb{R} \text{ for each } i,$$

where $\sigma_1, \cdots, \sigma_n$ are r -dimensional simplexes which are pairwise non-overlapping.

Suppose $A = a_1\sigma_1 + \cdots + a_n\sigma_n$, then the boundary ∂A of Polyhedron A can be defined by

$$\partial A = a_1\partial\sigma_1 + \cdots + a_n\partial\sigma_n.$$

In this paper, we let $\text{Vol}(\sigma)$ denote the volume of simplex σ , which is the r -dimensional Lebesgue measure of σ . For example, $\text{Vol}(pq) = |p - q|$ the length of the segment of pq , and $\text{Vol}(abc)$ is the area of triangle abc . If $A = a_1\sigma_1 + \cdots + a_n\sigma_n$ with $\sigma_1, \cdots, \sigma_n$ pairwise non-overlapping, the mass $M(A)$ of A is defined as

$$M(A) = |a_1| \cdot \text{Vol}(\sigma_1) + \cdots + |a_n| \cdot \text{Vol}(\sigma_n)$$

For example in Figure 3, we have two parallel directed simplexes S, T of length 1.

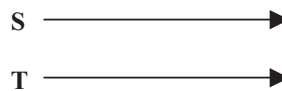


Figure 3: parallel simplexes S and T

Then $M(S) = 1$, and $M(T) = 1$ and

$$M(S - T) = \text{Vol}(S) + |-1| \cdot \text{Vol}(T) = 2$$

Flat norm and flat chain [7]

In the above figure, when S is fixed and move T such that T is close to S more and more, we can define a norm $|\cdot|$ such that $|T - S| \rightarrow 0$.

For polyhedron T , let flat Norm $|T|$ of T be defined by

$$|T| = \inf\{M(A) + M(B) : T = A + \partial B\}.$$

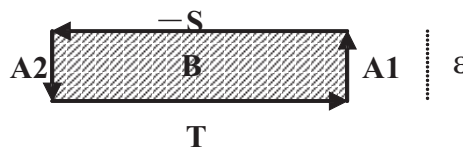


Figure 4: A circle composed of $-S, A2, T, A1$.

Let B be the solid square in the figure, then

$$-S + A2 + T + A1 = \partial B,$$

i.e., $T - S = -(A1 + A2) + \partial B$. Let $A = -(A1 + A2)$, then

$$T - S = A + \partial B,$$

where $M(A) = 2\varepsilon$ and area of B is ε , which means $M(B) = \varepsilon$. By the above definition, we have

$$|T - S| \leq M(A) + M(B) = 2\varepsilon + \varepsilon = 3\varepsilon \rightarrow 0.$$

We have following 4 property of flat norm:

- (1) $0 \leq |T| \leq M(T)$ (Let $T = T + \partial 0$);
- (2) $|kT| = |k| \cdot |T|$. (Since $M(kA) = |k|M(A)$);
- (3) $|S + T| \leq |S| + |T|$ (Since $M(A_1 + A_2) \leq M(A_1) + M(A_2)$);
- (4) $|T| = 0$ if and only if $T = 0$.

Proof of (4): It suffices to show that $T \neq 0$ implies $|T| > 0$.

Let $T = a_1\sigma_1 + \dots + a_n\sigma_n$, then we can construct non-negative C^∞ differential form ω supported on one of simplex such that

$$\int_T \omega > 0.$$

On the other hand, let $T = A + \partial B$. Because B is piecewise smooth, it follows from Stokes's formula that

$$\int_{\partial B} \omega = \int_B d\omega,$$

hence

$$\left| \int_T \omega \right| = \left| \int_A \omega + \int_{\partial B} \omega \right| \leq \left| \int_A \omega \right| + \left| \int_B d\omega \right| \leq M(A) \sup |\omega| + M(B) \sup |d\omega|.$$

Let $\|\omega\| = \sup(|\omega|, |d\omega|)$, then the above inequality shows that

$$\left| \int_T \omega \right| \leq [M(A) + M(B)] \cdot \|\omega\|$$

Takeing A and B such that $T = A + \partial B$ and $M(A) + M(B) \rightarrow |T|$, then we have

$$0 < \left| \int_T \omega \right| \leq |T| \cdot \|\omega\|$$

That means $|T| > 0$. \square

By the above 4 properties, the flat norm of polyhedron is a norm on the linear space spanning by r -dimensional polyhedron. Making completion of this linear space by flat norm to obtain a new space, we call every element of this new space r -dimensional flat chain.

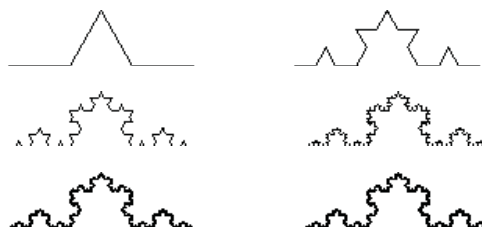


Figure 5: the Koch curve with A_1, A_2, \dots, A_6

Theorem 1 *The koch curve is a flat chain.*

Proof. Suppose the Koch curve ($\subset \mathbb{R}^2$) is generated by $\{S_i\}_{i=1}^4$ with $(0, 0)$ and $(1, 0)$ being fixed points of S_1 and S_4 respectively [1]. In fact,

$$S_1(x, y) = (x, y)/3, S_4(x, y) = (x, y)/3 + (2, 0)/3$$

and

$$\begin{aligned} S_1(1, 0) &= S_2(0, 0) = (1/3, 0), \\ S_2(1, 0) &= S_3(0, 0) = (1/2, \sqrt{3}/6), \\ S_3(1, 0) &= S_4(0, 0) = (2/3, 0). \end{aligned}$$

Let $K = [0, 1] \times \{0\}$. Equipping K with a direction from $(0, 0)$ and $(1, 0)$, we consider K as a flat chain. Let $\Omega_n = \{1, 2, 3, 4\}^n$ denote all the word of length n , and

$$A_n = \sum_{i_1 \dots i_n \in \Omega_n} S_{i_1 \dots i_n}(K).$$

Then for any n, m with $m > n$, $A_n - A_m$ is a boundary of some 2-dimensional chain, with its area less than $4^n(1/3)^{2n}(\sqrt{3}/6) = (4/9)^n(\sqrt{3}/6)$, i.e., for any n, m with $m > n$,

$$|A_n - A_m| \leq (4/9)^n(\sqrt{3}/6) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which implies $\{A_n\}_n$, with its limit being a flat chain. \square

2 Flat Cochain

Making completion of the linear space of r -dimensional polyhedrons, we get a space Π_r of all the r -dimensional flat chain with norm $|\cdot|$. A bounded linear operator X on Π_r is said to be a **flat cochain** [7] with is norm defined by

$$|X| = \sup_{|A|=1} |X \cdot A|.$$

We can define dX such that for any $(r + 1)$ -dimensional flat chain D ,

$$dX \cdot D = X \cdot (\partial D).$$

Then we have the following remarks:

(1) For any polyhedron T , $|\partial T| \leq |T|$

In fact, let $T = A + \partial B$ such that $M(A) + M(B) < |T| - \varepsilon$, then

$$\partial T = \partial A.$$

Therefore, $|\partial T| \leq M(A) \leq M(A) + M(B) < |T| - \varepsilon$. Letting $\varepsilon \rightarrow 0$, we get $|\partial T| \leq |T|$.

(2) Boundary operator ∂ can be defined on all the flat chains as follows:

Let S be a flat chain and polyhedrons $A_1, A_2, \dots, A_k, \dots$ approaches to S under the flat norm, i.e.,

$$|S - A_k| \rightarrow 0 \text{ as } k \rightarrow \infty, |A_{k_1} - A_{k_2}| \rightarrow 0 \text{ as } k_1, k_2 \rightarrow \infty.$$

Thus

$$|\partial A_{k_1} - \partial A_{k_2}| = |\partial(A_{k_1} - A_{k_2})| \leq |A_{k_1} - A_{k_2}| \rightarrow 0,$$

denoted by ∂S the limit of these Cauchy sequence $\{\partial A_k\}_k$. Extending the above inequality for all flat chains, we have

$$|\partial T| \leq |T|.$$

(3) $|dX| \leq |X|$, that means dX is a flat cochain.

Since $|dX \cdot A| = |X \cdot \partial A| \leq |X| \cdot |\partial A| \leq |X| \cdot |A|$, $|dX| = \sup_{|A|=1} |dX \cdot A| \leq \sup_{|A|=1} |X| \cdot |A| = |X|$,

$$|dX| \leq |X|.$$

For any cochain X , since $|A| \leq M(A)$, $|B| \leq M(B)$, we have

$$|X \cdot A| \leq |X| \cdot |A| \leq |X| \cdot M(A)$$

$$|dX \cdot B| \leq |dX| \cdot |B| \leq |X| \cdot M(B)$$

Lipschitz form and flat cochain

Lemma 1 Suppose $U \subset \mathbb{R}^n$ is open, and $K \subset U$ is compact. Let $f : U \rightarrow \mathbb{R}$ be a Lipschitz function. Assume that for any $x, y \in U$,

$$|f(x) - f(y)| \leq C|x - y|$$

where C is a constant, then for any $\varepsilon > 0$, there is a C^∞ function φ such that

$$\sup_{x \in K} |f(x) - \varphi(x)| < \varepsilon,$$

and for any $x, y \in K$,

$$|\varphi(x) - \varphi(y)| \leq C|x - y|,$$

Proof: Take a non-negative function h supported in the unit ball such that $\int h = 1$. Let $h_\rho = \rho^{-n}h(\frac{x}{\rho})$ and $\varphi_\rho = h_\rho * f$, that is $\varphi_\rho(x) = \int_{B(0,\rho)} h_\rho(z)f(x - z)dz$. Here $\partial^\alpha \varphi_\rho = \partial^\alpha h_\rho * f$, which implies φ_ρ is a C^∞ function.

$$\begin{aligned} |f(x) - \varphi_\rho(x)| &= \left| \int_{B(0,\rho)} h_\rho(z)[f(x) - f(x - z)]dz \right| \\ &\leq \left| \int_{B(0,\rho)} h_\rho(z) \right| \cdot \sup_{|z| < \rho} |f(x) - f(x - z)| \leq C\rho, \end{aligned}$$

and

$$|\varphi_\rho(x_1) - \varphi_\rho(x_2)| = \left| \int_{B(0,\rho)} h_\rho(z)[f(x_1 - z) - f(x_2 - z)]dz \right| \leq C|x_1 - x_2|.$$

□

Theorem 2 If $\omega = \sum_{i=1}^n f_i dx_i$ with f_i being Lipschitzian, then ω is a flat cochain.

proof. It suffices to show there exists a constant N such that for any $(r + 1)$ -dimensional simplex σ^{r+1} ,

$$\left| \int_{\partial\sigma^{r+1}} \omega \right| \leq N \text{Vol}(\sigma^{r+1}).$$

By the above lemma, we select differential form ω' such that

$$\|\omega - \omega'\|_K < \varepsilon,$$

and $\sup \|d\omega'\|_K < M$, where M only depends on ω .

For this $(r + 1)$ -simplex contained in the compact set K , Stokes's formula holds and thus

$$\left| \int_{\partial\sigma^{r+1}} \omega' \right| = \left| \int_{\sigma^{r+1}} d\omega' \right| \leq M \cdot \text{Vol}(\sigma^{r+1}).$$

Since $\|\omega - \omega'\|_K < \varepsilon$, $\left| \int_{\partial\sigma^{r+1}} \omega' \right| \rightarrow \left| \int_{\partial\sigma^{r+1}} \omega \right|$. As $\varepsilon \rightarrow 0$, the above inequality implies

$$\left| \int_{\partial\sigma^{r+1}} \omega \right| \leq M \cdot \text{Vol}(\sigma^{r+1}).$$

This completes the proof of theorem. □

3 Conclusion

Since the Koch curve γ is a 1-dimensional flat chain, and any 1-dimensional Lipschitz form $\omega = f dx + g dy$ is a flat cochain, and thus $\int_\gamma \omega$ is well defined.

Acknowledgements

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