

Peakon and Compacton Solutions for $K(p,q,1)$ Equation

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Abstract: The variational iteration method is used for solving two types of nonlinear partial differential equations such as $K(2,2,1)$ equation and $K(3,3,1)$ equation, the chosen initial solution can be in compacton form or peakon form with some unknown parameters which can be determined in the solution procedure.

Key words: $K(p,q,1)$ equation; variational iteration method; compacton

1 Introduction

In this paper, we consider the following nonlinear dispersive equations $K(p,q,1)$

$$u_t + a(u^p)_x + (u^q)_{3x} + u_{5x} = 0 \tag{1}$$

Compacton solutions to Eq. (1) have been found by sine-cosine method[1] and Adomian decomposition method[2]. The motivation of this paper is to extend the analysis of the variational iteration method to solve Eq. (1), and we also want to find peakon solutions to Eq. (1).

The variational iteration method has been shown [3-5] to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions. More great interest has been shown in special solutions[6-8].

In the next section, we give the correction functional to $K(p,q,1)$ equation. Particularly, peakon solutions and compacton solutions to $K(2,2,1)$ equation or $K(3,3,1)$ equation are found by variational iteration method.

2 Exact solutions by variational iteration method

In this section, we will use the variational iteration method[3-5] to solve Eq. (1). And we consider the correction functional as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \{ (u_n)_t + a(\tilde{u}_n^p)_x + (\tilde{u}_n^q)_{3x} + (\tilde{u}_n)_{5x} \} d\tau = 0 \tag{2}$$

where λ is a general Lagrange multiplier, and \tilde{u}_n denotes restricted variation, i.e. $\delta \tilde{u}_n = 0$.

With the help of the above correction functional stationary, we obtain the following stationary conditions

$$\lambda'(\tau) = 0 \tag{3}$$

$$1 + \lambda(\tau)|_{t=\tau} = 0 \tag{4}$$

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The Lagrangian multiplier, therefore, can be identified as

$$\lambda = -1 \quad (5)$$

Substituting Eq.(5) into the correction functional Eq. (2) results in the following formula

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{(u_n^p)_t + a(u_n^p)_x + (u_n^q)_{3x} + (u_n)_{3x}\} d\tau \quad (6)$$

As illustrating examples, we only consider K(2, 2,1) and K(3, 3,1) in the following sections.

2.1 Compacton and peakon solutions to K(2, 2,1) equation

Considering K(2, 2,1) equation:

$$u_t + a(u^2)_x + (u^2)_{3x} + u_{5x} = 0 \quad (7)$$

And (7)' correction iteration formula as

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t (u_n)_t + a(u_n^2)_x + ((u_n)^2)_{3x} + (u_n)_{5x} d\tau \quad (8)$$

To search for its compacton solution, we assume the solution of the form

$$u_0(x, t) = A \cos^2 k(x + dt) \quad (9)$$

where A, k, d are constants to be determined. Substituting (9) into (8), and let

$$u_n(x, t) = u_{n+1}(x, t), \quad \frac{\partial^k}{\partial t^k} u_n(x, t) = \frac{\partial^k}{\partial t^k} u_{n+1}(x, t) \quad (10)$$

With the help of Mathematica, we can obtain

$$\frac{\partial}{\partial t} u_0(x, t) = -2Adk \cos k(x + dt) \sin k(x + dt) \quad (11)$$

$$\begin{aligned} \frac{\partial}{\partial t} u_1(x, t) &= 24aA^2k^3 \cos k(x + dt) \sin k(x + dt) + 32Ak^5 \cos k(x + dt) \sin k(x + dt) \\ &+ 4A^2k \cos^3 k(x + dt) \sin k(x + dt) - 64aA^2k^3 \cos^3 k(x + dt) \sin k(x + dt) \end{aligned} \quad (12)$$

Setting $\frac{\partial}{\partial t} u_0(x, t) = \frac{\partial}{\partial t} u_1(x, t)$, we can obtain

$$\begin{aligned} -2Adk \cos k(x + dt) &= (24aA^2k^3 + 32Ak^5 - 2Adk) \cos k(x + dt) \\ &+ (4A^2k - 64aA^2k^3) \cos^3 k(x + dt) \end{aligned} \quad (13)$$

Equating the coefficients of like power of $\cos k(x + dt)$ yields

$$-2Adk = 24aA^2k^3 + 32Ak^5 - 2Adk \quad (14)$$

$$4A^2k - 64aA^2k^3 = 0 \quad (15)$$

From (14)–(15), we can obtain

$$A = \frac{-1 - 16a^2d}{12a^2}, k = \pm \frac{1}{4\sqrt{a}} \quad (16)$$

Substituting (16) into (9), we can obtain compacton solutions as

$$u(x, t) = \frac{-1 - 16a^2d}{12a^2} \cos^2 \frac{\pm 1}{4\sqrt{a}}(x + dt) \tag{17}$$

We can also begin with a more general initial solution in the form

$$u_0(x, t) = A \exp k(x + dt) \tag{18}$$

Setting $\frac{\partial}{\partial t} u_0(x, t) = \frac{\partial}{\partial t} u_1(x, t)$, with the help of Mathematica, we can obtain

$$Akd e^{k(x+dt)} = (-2A^2k - 8aA^2k^3)e^{2k(x+dt)} - Ak^5 e^{k(x+dt)} \tag{19}$$

From (19), we can obtain

$$d = -\frac{1}{16a^2}, \quad k = \pm \frac{1}{2\sqrt{-a}} \tag{20}$$

where A is an arbitrary constant and $a < 0$, Substituting (20) into (18), we can obtain peakon solutions as

$$u(x, t) = A e^{-\frac{1}{2\sqrt{-a}} x - \frac{1}{16a^2} t} \tag{21}$$

2.2 Compacton and peakon solutions to $K(3, 3, 1)$ equation

Consider $K(3, 3, 1)$ equation:

$$u_t + a(u^3)_x + (u^3)_{3x} + u_{5x} = 0 \tag{22}$$

and its correction iteration formula as

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t (u_n)_t + a(u_n^3)_x + (u_n^3)_{3x} + (u_n)_{5x} d\tau = 0 \tag{23}$$

Assuming an initial condition as

$$u_0(x, t) = A \cos k(x + dt) \tag{24}$$

where A, k, d are constants to be determined. Substituting (24) into (23), and let

$$u_n(x, t) = u_{n+1}(x, t), \quad \frac{\partial^k}{\partial t^k} u_n(x, t) = \frac{\partial^k}{\partial t^k} u_{n+1}(x, t) \tag{25}$$

With the help of Mathematica, we can obtain

$$\frac{\partial}{\partial t} u_0(x, t) = -Adk \sin k(x + dt) \tag{26}$$

$$\begin{aligned} \frac{\partial}{\partial t} u_1(x, t) = & 6aA^2k^3 \sin k(x + dt) + Ak^5 \sin k(x + dt) + \\ & 3A^3k \cos^2 k(x + dt) \sin k(x + dt) - 27aA^3k^3 \cos^2 k(x + dt) \sin k(x + dt) \end{aligned} \tag{27}$$

Setting $\frac{\partial}{\partial t} u_0(x, t) = \frac{\partial}{\partial t} u_1(x, t)$, we can obtain

$$-Adk = 6aA^2k^3 + Ak^5 + 3A^3k \cos^2 k(x + dt) - 27aA^3k^3 \cos^2 k(x + dt) \tag{28}$$

Equating the coefficients of like power of $\cos k(x + dt)$ yields

$$-Adk = 6aA^2k^3 + Ak^5 \tag{29}$$

$$3A^3k - 27aA^3k^3 = 0 \tag{30}$$

From (29)–(30), we can obtain

$$A = \frac{-1 - 81a^2d}{54a^2}, k = \pm \frac{1}{3\sqrt{a}} \quad (31)$$

Substituting (31) into (24), we can obtain compacton solutions as

$$u(x, t) = \pm \sqrt{\frac{-1 - 81a^2d}{54a^2}} \cos \frac{\pm 1}{3\sqrt{a}}(x + dt) \quad (32)$$

We can also begin with a more general initial solution in the form

$$u_0(x, t) = A \exp k(x + dt) \quad (33)$$

Setting $\frac{\partial}{\partial t} u_0(x, t) = \frac{\partial}{\partial t} u_1(x, t)$, with the help of Mathematica, we can obtain

$$Akd e^{k(x+dt)} = (-3A^3k - 27aA^3k^3)e^{3k(x+dt)} - Ak^5 e^{k(x+dt)} \quad (34)$$

From (34), we can obtain

$$d = -\frac{1}{81a^2}, \quad k = \pm \frac{1}{3\sqrt{-a}} \quad (35)$$

where A is an arbitrary constant and $a < 0$, Substituting (35) into (33), we can obtain peakon solutions as

$$u(x, t) = A e^{-\frac{1}{3\sqrt{-a}}x - \frac{1}{81a^2}t} \quad (36)$$

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