

Adaptive Synchronization of an Uncertain Complex Delayed Dynamical Networks

Zhengquan Yang *, Zhongxin Liu, Zengqiang Chen, Zhuzhi Yuan
Department of Automation Nankai University
Tianjin, 300071, China

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Abstract: This paper proposes some locally and globally adaptive synchronization criterias for uncertain complex delayed dynamical networks. For complex network with unknown dynamics of nodes and unknown couplings including uniform and nonuniform inner couplings, some simple linear feedback controllers with updated strength are designed. It is useful for future practical engineering design. Furthermore, numerical simulations are presented to illustrate the effectiveness of these synchronization criterias.

Key words: complex networks; synchronization; adaptive synchronization; time delay system; chaotic system

1 Introduction

Complex networks have been intensively investigated in various disciplines over the last decade [1-8]. A complex network is a large set of interconnected nodes, where the nodes and connections can represent everything. Detailed examples are the World Wide Web, electrical power grids, Internet, food webs, communication networks, and so on.

Over the last few years, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes in a network [5-12]. In fact, synchronization is a kind of typical collective behaviors and basic motions in nature. For example, the synchronization of coupled oscillators can well explain many natural phenomena.

Systems with time delays are quite ubiquitous in nature. The time delays are usually caused by finite signal transmission speeds and memory effects. Due to the finite speeds of transmission and spreading as well as traffic congestions, a signal or influence traveling through a network is often associated with time delays, and this is very common in biological and physical networks. Therefore, time delays should be modeled in order to simulating more realistic networks. In [13] C. Li and G. Chen introduced complex dynamical network models with coupling delays for both continuous and discrete-time cases and investigate their synchronization phenomena and criteria. In [14] J. Zhou and T. Chen investigated synchronization dynamics of a general model of complex delayed networks as well as the effects of time delays.

In these investigations, an essential requirement is that the structure of the network and the coupling functions are known a priori. Notice also that the final state of the network will reach after achieving synchronization is usually not known beforehand and cannot be changed at future. Moreover, one cannot guarantee that all linearly coupled dynamical nodes can synchronize if the network size is sufficient large for some nearest-neighboring coupled networks. Yet, it is very desirable if one can choose what state to which the network will synchronize. Likewise, it is preferable if the coupling functions in a dynamical network

* **Corresponding author.** E-mail address: yzquan@mail.nankai.edu.cn

are not restricted to be linear and completely known, since in applications quite often this requirement is not realistic. Sometimes, even the network structure is only partially known or completely unknown. If there are too many unknown issues, of course, it is very difficult or even impossible to design a controller to achieve the intended network synchronization; however, it would be interesting to see if the task is possible for a general situation where the network coupling is an unknown but bounded nonlinear function, where the bounds may be either known or unknown.

In this paper, several novelty locally and globally asymptotic stable network synchronization criterias are deduced based on Lyapunov stability theory for the uncertain complex dynamical networks. Especially, our sufficient conditions for network synchronization are rather widely and the controllers are very simple. It is very useful for future practical engineering design.

This paper is organized as follows. An uncertain complex dynamical network model and several necessary hypotheses are given in Section II. In Section III, locally and globally adaptive synchronization criteria for uncertain complex dynamical networks are proposed. In Section IV, a simulation example is provided to verify the effectiveness of the proposed method. Finally, conclusions are given in Section V.

2 Preliminaries

This section introduces an uncertain complex dynamical network model and gives some preliminary definitions and hypotheses. Consider an uncertain complex dynamical network consisting of N identical nonlinear oscillators with uncertain nonlinear diffusive coupling, which is described by

$$\dot{x}_i = f(x_i, t) + h_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau)) + u_i. \quad (1)$$

where $1 \leq i \leq N$, $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in R^n$ is the state vector of the i th node, $f : \Omega \times R^+ \rightarrow R^n$ is a smooth nonlinear vector function, node dynamics is $\dot{x} = f(x, t)$, $h_i : \Omega \times R^+ \rightarrow R^n$ are unknown nonlinear smooth diffusive coupling functions, $u_i \in R^n$ are the control inputs, and the coupling-control terms satisfy $h_i(x_1, x_2, \dots, x_N) + u_i = 0$,

Let $C = C([- \tau, 0], R^n)$ be the Banach space of continuous function mapping the interval $[- \tau, 0]$ into R^n with the norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. For the nonautonomous functional differential equations (1), its initial condition is an element of $R \times C^N$. Under some condition on $f, h_i (i = 1, 1, \dots, N)$, for any given $\phi = (\phi_1, \phi_2, \dots, \phi_N) \in C^N$ with $\phi_i \in C (i = 1, 2, \dots, N)$ and $t_0 \in R$, there exist a unique global continuous solution of (1) that is defined as $x_i(t, t_0, \phi)$ for all $t > t_0 - \tau (i = 1, 2, \dots, N)$. For simplicity, we denote $x_i(t, t_0, \phi)$ by $x_i(t) (i = 1, 2, \dots, N)$

Network synchronization is a typical collective behavior. In the following rigorous mathematical definition is introduced for the concept of network synchronization.

Assume that $s(t)$ is a solution of node system $\dot{s}(t) = f(s(t), t)$. Then $S(t) = (s^T(t), s^T(t), \dots, s^T(t))^T$ is a synchronous solution of uncertain dynamical network (1) since it is a diffusive coupling network. Here, $s(t)$ can generate an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

Define error vector

$$e_i(t) = x_i(t) - s(t), \quad 1 \leq i \leq N \quad (2)$$

Then the objective of controller u_i is to guide the dynamical network (1) to synchronize. That is,

$$\lim_{t \rightarrow \infty} \|e_i(t)\|_2 = 0, \quad 1 \leq i \leq N. \quad (3)$$

Since $\dot{s} = f(s, t)$, from network (1), we have

$$\dot{e}_i = \bar{f}(x_i, s, t) + \bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau), s) + u_i \quad (4)$$

where $\bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau), s) = h_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau)) - h_i(s, s, \dots, s)$, $\bar{f}(x_i, s, t) = f(x_i, t) - f(s, t)$, $1 \leq i \leq N$.

In the following, we give several useful hypotheses.

Hypothesis 1 (H1) Assume that there exists a nonnegative constant α satisfying $\|Df(s, t)\|_2 = \|A(t)\|_2 \leq \alpha$, where $A(t)$ is the Jacobian of $f(s, t)$ on $s(t)$.

Hypothesis 2 (H2) Suppose that there exist a set of nonnegative constants γ_{ij} ($1 \leq i, j \leq N$), satisfying $\|\bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau), s)\|_2 \leq \sum_{j=1}^N \gamma_{ij} \|e_j(t - \tau)\|_2$, for $1 \leq i \leq N$.

Remark 1 If H1 holds, then we get $\|\frac{A(t)+A^T(t)}{2}\|_2 \leq \alpha$.

3 Adaptive synchronization of an uncertain complex delayed dynamical network

This section discusses the local synchronization and global synchronization of the uncertain complex delayed dynamical network (1). Several network synchronization criteria are given.

3.1 Local Synchronization

Linearizing error system (4) at zero gives

$$\dot{e}_i = A(t)e_i(t) + \bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau), s) + u_i \quad (5)$$

where $1 \leq i \leq N$ and recall that $A(t) = Df(s, t)$ is the Jacobian of f evaluated at $x = s(t)$.

Based on H1 and H2, a network synchronization criterion is deduced as follows.

Theorem 1 Suppose that H1 and H2 hold. Then the synchronous solution $S(t)$ of uncertain dynamical network (1) is locally asymptotic stable under the adaptive controllers

$$u_i = -d_i e_i, \quad 1 \leq i \leq N \quad (6)$$

and updating laws

$$\dot{d}_i = k_i e_i^T e_i = k_i \|e_i\|_2^2, \quad 1 \leq i \leq N, \quad (7)$$

where k_i ($1 \leq i \leq N$) are positive constants

Proof: Consider the Lyapunov candidate as follows:

$$V = \frac{1}{2} \sum_{i=1}^N [e_i^T e_i + \int_{t-\tau}^t e_i^T(r) e_i(r) dr] + \frac{1}{2} \sum_{i=1}^N \frac{(d_i - \hat{d}_i)^2}{k_i}. \quad (8)$$

where $\hat{d}_i (\leq i \leq N)$ are positive constants to be determined. Thus one gets

$$\begin{aligned}
\dot{V} &= \frac{1}{2} \sum_{i=1}^N [\dot{e}_i^T e_i + e_i^T \dot{e}_i + e_i^T e_i - e_i^T(t-\tau)e_i(t-\tau)] + \sum_{i=1}^N \frac{(d_i - \hat{d}_i)\dot{d}_i}{k_i} \\
&= \sum_{i=1}^N e_i^T \left(\frac{A(t) + A^T(t)}{2} - d_i I_n + \frac{1}{2} I_n \right) e_i + \sum_{i=1}^N e_i^T \bar{h}_i(x_1(t-\tau), x_2(t-\tau), \dots, x_N(t-\tau), s) \\
&\quad - \frac{1}{2} \sum_{i=1}^N e_i^T(t-\tau)e_i(t-\tau) + \sum_{i=1}^N (d_i - \hat{d}_i) e_i^T e_i \\
&\leq \sum_{i=1}^N e_i^T \left(\frac{A(t) + A^T(t)}{2} - \hat{d}_i I_n + \frac{1}{2} I_n \right) e_i + \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \|e_i\|_2 \|e_j(t-\tau)\|_2 \\
&\quad - \frac{1}{2} \sum_{i=1}^N e_i^T(t-\tau)e_i(t-\tau) \\
&\leq \sum_{i=1}^N (\alpha - \hat{d}_i + \frac{1}{2}) \|e_i\|_2^2 + \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \|e_i\|_2 \|e_j(t-\tau)\|_2 - \frac{1}{2} \sum_{i=1}^N \|e_i^T(t-\tau)\| \|e_i(t-\tau)\| \\
&= (\alpha - \hat{d}_i + \frac{1}{2}) e^T e + e^T \Gamma e(t-\tau) - \frac{1}{2} e(t-\tau)^T e(t-\tau) \\
&\leq (\alpha - \hat{d}_i + \frac{1}{2}) e^T e + \frac{1}{2} e^T \Gamma^T \Gamma e \\
&\leq e^T (\text{diag}\{\alpha - \hat{d}_1 + \frac{1}{2}, \alpha - \hat{d}_2 + \frac{1}{2}, \dots, \alpha - \hat{d}_N + \frac{1}{2}\} + \frac{1}{2} \Gamma^T \Gamma) e
\end{aligned} \tag{9}$$

where $e = (\|e_1\|_2, \|e_2\|_2, \dots, \|e_N\|_2)^T$, $e(t-\tau) = (\|e_1(t-\tau)\|_2, \|e_2(t-\tau)\|_2, \dots, \|e_N(t-\tau)\|_2)^T$ and $\Gamma = (\gamma_{ij})_{N \times N}$.

Since α and $\gamma_{ij} (1 \leq i \leq N)$ are nonnegative constants, one can select suitable constants $\bar{d}_i (1 \leq i \leq N)$ to make $\text{diag}\{\alpha - \hat{d}_1 + \frac{1}{2}, \alpha - \hat{d}_2 + \frac{1}{2}, \dots, \alpha - \hat{d}_N + \frac{1}{2}\} + \frac{1}{2} \Gamma^T \Gamma$ be a negative definite matrix. Thus it follows error vector $\eta = (e_1^T, e_2^T, \dots, e_N^T)^T \rightarrow 0$ as $t \rightarrow +\infty$. That is, the synchronous solution $S(t)$ of uncertain dynamical network (1), The proof is thus completed.

Assume that the coupling of network (1) is linear satisfying $h_i(x_1(t-\tau), x_2(t-\tau), \dots, x_N(t-\tau)) = \sum_{j=1}^N b_{ij} x_j(t-\tau)$ for $1 \leq i \leq N$, where $b_{ij} (1 \leq i \leq N)$ are constants. Then uncertain network (1) is recasted as follows:

$$\dot{x}_i = f(x_i, t) + \sum_{j=1}^N b_{ij} x_j(t-\tau) + u_i, \quad 1 \leq i \leq N \tag{10}$$

For linear coupling, H2 is naturally satisfied. Thus one gets the following corollaries.

Corollary 1 Suppose that H1 holds. Thus the synchronous solution $S(t)$ of uncertain dynamical network (10) is locally asymptotic stable under the adaptive controllers (6) and updating laws (7).

Moreover, for coupling scheme $h_i(x_1(t-\tau), x_2(t-\tau), \dots, x_N(t-\tau)) = \sum_{j=1}^N b_{ij} p(x_j(t-\tau))$ with $1 \leq i \leq N$, where $b_{ij} (1 \leq i \leq N)$ are constants satisfying $\sum_{j=1}^N b_{ij} = 0$ for $1 \leq i \leq N$ and $\|Dp(\xi)\|_2 \leq \delta\xi$ for $\xi \in \Omega$, where $\|Dp(\xi)\|$ is the Jacobian of $p(x)$ on ξ . the network (1) is rewrote as follows:

$$\dot{x}_i = f(x_i, t) + \sum_{j=1}^N b_{ij} p(x_j(t-\tau)) + u_i, \quad 1 \leq i \leq N \tag{11}$$

If H1 holds, then one has

$$\begin{aligned}
\bar{h}_i(x_1(t-\tau), x_2(t-\tau), \dots, x_N(t-\tau)) &= \sum_{j=1}^N b_{ij} [p(x_j(t-\tau)) - p(s(t-\tau))] \\
&\leq \sum_{j=1}^N \delta b_{ij} \|e_j(t-\tau)\|_2
\end{aligned} \tag{12}$$

for $1 \leq i \leq N$. That is, H2 holds and one gets the following corollary

Corollary 2 Assume that H1 holds. Then, under the adaptive controllers (6) and updating laws (7), the synchronous solution $S(t)$ of uncertain dynamical network (12) is locally asymptotic stable.

In the following subsection, we discuss the global synchronization case.

3.2 Global synchronization

This section presents two global network synchronization criterions.

Rewrite node dynamics $\dot{x}_i = f(x_i, t)$ as $\dot{x}_i = Bx_i(t) + g(x_i, t)$, where $B \in R^{n \times n}$ is a constant matrix, $g : \Omega \times R^+ \rightarrow R^n$ is a smooth nonlinear function. Thus network (1) is described by

$$\dot{x}_i = Bx_i + g(x_i, t) + h_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau)) + u_i. \tag{13}$$

where $1 \leq i \leq N$. Similarly, one can get the error system

$$\dot{e}_i = Be_i + \bar{g}(x_i, t) + \bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau)) + u_i. \tag{14}$$

where $1 \leq i \leq N$ and $\bar{g}(x_i, t) = g(x_i, t) - g(s, t)$.

Hypothesis 3 (H3) Suppose that there exists a nonnegative constant satisfying $\|\bar{g}(x_i, s, t)\|_2 \leq \mu \|e_i\|$

Thus one can get the following global network synchronization criterion.

Theorem 2 Suppose that H2 and H3 hold. Then the synchronous solution $S(t)$ of uncertain dynamical network (1) is locally asymptotic stable under the adaptive controllers

$$u_i = -d_i e_i, \quad 1 \leq i \leq N \tag{15}$$

and updating laws

$$\dot{d}_i = k_i e_i^T e_i = k_i \|e_i\|_2^2, \quad 1 \leq i \leq N, \tag{16}$$

where $k_i (1 \leq i \leq N)$ are positive constants

Proof: Since B is a given constant matrix, there exists a nonnegative constant β such that $\|B\|_2 \leq \beta$. It follows $\|\frac{B+B^T}{2}\|_2 \leq \beta$. Similarly, construct Lyapunov function (8), then one has

$$\begin{aligned} \dot{V} &= \sum_{i=1}^N e_i^T \left(\frac{A(t) + AT(t)}{2} - d_i I_n + \frac{1}{2} I_n \right) e_i - \frac{1}{2} \sum_{i=1}^N e_i^T(t - \tau) e_i(t - \tau) + \sum_{i=1}^N e_i^T \bar{g}_i(x_i, s, t) \\ &\quad + \sum_{i=1}^N e_i^T \bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau), s) + \sum_{i=1}^N (d_i - \hat{d}_i) e_i^T e_i \\ &\leq \sum_{i=1}^N (\beta + \mu - \hat{d}_i + \frac{1}{2}) \|e_i\|_2^2 + \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \|e_i\|_2 \|e_j(t - \tau)\|_2 - \frac{1}{2} \sum_{i=1}^N \|e_i^T(t - \tau)\| \|e_i(t - \tau)\| \\ &\leq e^T (\text{diag}\{\beta + \mu - \hat{d}_1 + \frac{1}{2}, \beta + \mu - \hat{d}_2 + \frac{1}{2}, \dots, \beta + \mu - \hat{d}_N + \frac{1}{2}\} + \frac{1}{2} \Gamma^T \Gamma) e \end{aligned} \tag{17}$$

where $e = (\|e_1\|_2, \|e_2\|_2, \dots, \|e_N\|_2)^T$, $e(t - \tau) = (\|e_1(t - \tau)\|_2, \|e_2(t - \tau)\|_2, \dots, \|e_N(t - \tau)\|_2)^T$ and $\Gamma = (\gamma_{ij})_{N \times N}$.

Since β, μ and $\gamma_{ij} (1 \leq i \leq N)$ are nonnegative constants, one can select suitable constants $\bar{d}_i (1 \leq i \leq N)$ to make $\text{diag}\{\beta + \mu - \hat{d}_1 + \frac{1}{2}, \beta + \mu - \hat{d}_2 + \frac{1}{2}, \dots, \beta + \mu - \hat{d}_N + \frac{1}{2}\} + \frac{1}{2} \Gamma^T \Gamma$ be a negative definite matrix. Thus it follows error vector $\eta = (e_1^T, e_2^T, \dots, e_N^T)^T \rightarrow 0$ as $t \rightarrow +\infty$. That is, the synchronous solution $S(t)$ of uncertain dynamical network (1). The proof is thus completed.

Similarly, one gets the following two corollaries of global network synchronization.

Corollary 3 Suppose that H3 holds. Then the synchronous solution $S(t)$ of uncertain linearly coupled dynamical network (10) is globally asymptotic stable under the adaptive controllers (15) and updating laws (16).

Corollary 4 Suppose that H3 holds. Then the synchronous solution $S(t)$ of uncertain dynamical network (11) is globally asymptotic stable under the adaptive controllers (15) and updating laws (16).

Hypothesis 4 (H4) Assume that $g(x, t)$ satisfies Lipschitz condition. That is, there exists a positive constant satisfying $\|g(x, t) - g(y, t)\| \leq \kappa \|x - y\|$, where κ is Lipschitz constant. Obviously, H4 implies H3. Now one has the following synchronization criterion.

Theorem 3 Suppose that H2 and H4 hold. Then the synchronous solution $S(t)$ of uncertain dynamical network (1) is globally asymptotic stable under the adaptive controllers (13) and updating laws (14).

4 Simulation

This section presents an example to show the effectiveness of above synchronization criterions. Consider a dynamical network consisting of 50 identical Lorenz systems. Here, node dynamics is described by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = A \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} \quad (18)$$

where

$$A = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$a = 10, b = \frac{8}{3}, c = 28$ and $1 \leq i \leq N$. And the network system is defined as follows:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = A \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} + \begin{pmatrix} f_1(x_i) - 2f_1(x_{i+1}) + f_1(x_{i+2}) \\ 0 \\ f_2(x_i) - 2f_2(x_{i+1}) + f_2(x_{i+2}) \end{pmatrix} + u_i \quad (19)$$

and

$$\dot{d}_i = k_i e_i^T e_i = k_i \|e_i\|_2^2 \quad (20)$$

$f_1(x_i) = a(x_{i2} - x_{i1}), f_2(x_i) = x_{i1}x_{i2} - bx_{i3}, x_{51} = x_1, x_{52} = x_2$, and $1 \leq i \leq N$

Since Lorenz attractor is bounded in a bounded region $\Phi \subset R^3$, there exists a constant M satisfying $|x_{ij}|, |s_j| \leq M$ for $1 \leq i \leq N$ and $j = 1, 2, 3$. Therefore, $\|g(x_i, s, t)\|_2 = \sqrt{(x_{i3}e_{i1} + s_1e_{i3})^2 + (x_{i2}e_{i1} + s_1e_{i2})^2} \leq 2M\|e_i\|_2$. Similar to [15], one has $\bar{h}_i(x_1(t - \tau), x_2(t - \tau), \dots, x_N(t - \tau)) \leq 3\sqrt{2(a^2 + M^2)}(\|e_i(t - \tau)\|_2 + \|e_{i+1}(t - \tau)\|_2 + \|e_{i+2}(t - \tau)\|_2)$

Thus H2 and H3 hold. According to Theorem 2, the synchronous solution $S(t)$ of dynamical network (19)-(20) is globally asymptotic stable.

Assume that $\kappa = 10, d_i(0) = 1, x_i(0) = (4 + 0.5i, 5 + 0.5i, 6 + 0.5i)$ for $1 \leq i \leq N$ and $s(0) = (4, 5, 6)$. The synchronous error e_i is shown in Fig. 1. Obviously, the zero is globally asymptotic stable for dynamical network (19)-(20).

5 Conclusions

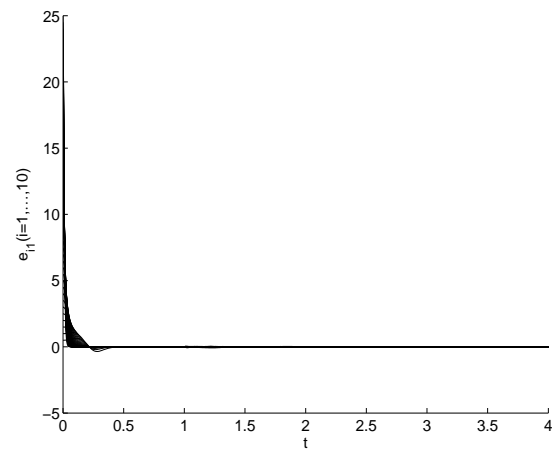
This paper presents some results on the locally and globally adaptive synchronization of an uncertain complex delayed dynamical network. Several novel network synchronization criterions are proved using Lyapunov stability theory. For complex network with unknown dynamics of nodes and unknown couplings including uniform and nonuniform inner couplings, some simple linear feedback controllers with updated strength are designed. It is useful for future practical engineering design. Furthermore, the effectiveness of these synchronization criterions are verified by numerical simulations.

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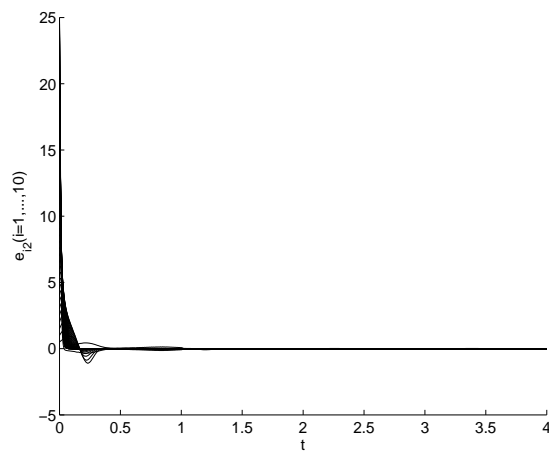
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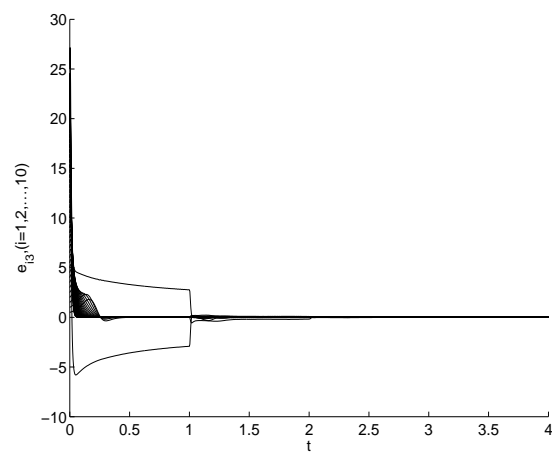
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(a)



(b)



(c)

Figure 1: Synchronization errors for the delayed network with time-delay $\tau = 1$.