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Explicit Solitary Solutions to the Toda Lattice Equation

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Abstract: A constructive method for exactly solving difference-difference equations(DDE) is presented, which is called the modified F-function method. In this method, the solution to DDEs is supposed to be a polynomial of the function F whereas F satisfies a general Riccati equation. By selecting proper coefficients of the Riccati equation, explicit solitary solutions of DDEs can be obtained. The Toda lattice equations is given as an example.

Keywords: differential-difference equation; modified F-function method; explicit solution; Riccati equation; Toda Lattice

1 Introduction

Since the famous work of Fermi, Pasta and Ulam in considering nonlinear lattice equation, nonlinear differential-difference equations (DDEs) have played a significant role in nonlinear science[1]. DDEs , which are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous paly an important role in modeling such physical phenomena as particle vibrations in lattices, charge fluctuations in network, pulses in biological chains, etc. One can say that DDEs occur whenever discrete phenomena are studied.

In the past several decades, more attentions were paid to study continuous solutions to ordinary differential equations (ODEs) and partial differential equations (PDEs)[2–5]. In the discrete literature, some powerful methods have been presented such as, inverse method, Backlund transformation, Darboux transformation, Hirota bilinear method, etc.[6, 7]. In 2004, Baldwin et al. [1] extended the continuous tanh function method to nonlinear DDEs and obtained solitary wave solutions (hyperbolic tangent functions) of some discrete soliton equations. Since then, many modified method have appeared , for example, the sine-Gordon expansion method, modified hyperbolic function method,etc[8–10] Their work allows one to directly derive many different types of travelling wave solutions of DDEs.

To seek for the different solutions of DDEs, a natural question is whether we can extend those methods in continuous equations to the discrete counterpart. In this paper, we would like to consider a modified method to solve nonlinear differential-difference equations.

2 Summaryof the modified F-method for DDEs

We would like to describe the modified F-method to solve nonlinear polynomial DDEs. The main idea is to seek solution of DDE by solving an algebraic system. Consider the following 1+1 dimensional nonlinear

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polynomial DDE,

$$\triangle(u_{n+p_1}(t), u_{n+p_2}(t), \cdots, u_{n+p_k}(t), u'_{n+p_1}(t), u'_{n+p_2}(t), ..., u'_{n+p_k}(t), \cdots, u^{(r)}_{n+p_1}(t), u^{(r)}_{n+p_2}(t), \cdots, u^{(r)}_{n+p_k}(t) = 0, \quad (1)$$

where the dependent variable u_n is a function of the varible n and the time variable t. t is continous while $n, p_i \in Z$ .And $u^{(r)}(t)$ denotes derivative terms of order r.

We seek solutions in the traveling frame of reference,

$$\xi_n = dn + ct + \xi_0, \quad (2)$$

where the coefficients d, c and the phase ξ_0 are all constants.

Step 1. Take the transformation
In the modified F-method, we assume solution to (1) has the polynomial form

$$u_n(t) = U(\xi_n) = a_0 + \sum_{i=-N}^{i=N} a_i F_n^i(\xi_n) \quad (3)$$

where a_i is a constant , N is an integer and $F_n(\xi_n)$ satifies the Ricaati equation

$$\frac{dF_n(\xi_n)}{d\xi_n} = p + qF_n(\xi_n) + rF_n^2(\xi_n), \quad (4)$$

where p, q, and r are all constants and $r \neq 0$. It should be noticed that for any s, $(s = 1, \cdots, k)$, u_{n+p_s} is a function of F_n and not F_{n+p_s}.

Using (3) and repeatedly applying the chain rule

$$\frac{d\bullet}{dt} = \frac{d\bullet}{dF_n}\frac{dF_n}{d\xi_n}\frac{d\xi_n}{dt} = c(p + qF_n(\xi_n) + rF_n^2(\xi_n))\frac{d\bullet}{dF_n}, \quad (5)$$

change (1) into

$$P(U_{n+p_1}(F_n), U_{n+p_2}(F_n), \cdots, U_{n+p_k}(F_n), U'_{n+p_1}(F_n), U'_{n+p_2}(F_n), ..., U'_{n+p_k}(F_n), \cdots, U^{(r)}_{n+p_1}(F_n), U^(r)_{n+p_2}(F_n), \cdots, U^{(r)}_{n+p_k}(F_n) = 0, \quad (6)$$

where $'$ denotes derivative about ξ_n. Balancing the highest derivative term and the nonlinear terms , we can determine the integer N in (3). To determine the degree of the polynomial solutions, we can substitute only the leading term.

Let's see the solution of (4). Different coefficents in (4) lead to different forms of F_n, as can be seen in Table 1. One can see that if $p = 1, q = 0, r = -1$ and choseing $F_n = tanh(\xi_n)$, it is the tanh method used in [1]. Case i and Case ii are the method used by Xie[8]. While other cases have not be seen before as limited to our knowledge.

Notice that

$$tanh(x+y) = \frac{tanh(x) + tanh(y)}{1 + tanh(x)tanh(y)}, \quad coth(x+y) = \frac{coth(x) + tanh(y)}{1 + coth(x)tanh(y)} \quad (7)$$

For Case i, we can rewrite the expressions (7) in a uniform formula by using of the expression

$$F(\xi_n + y) = \frac{F(\xi_n) + tanh(y)}{1 + F(\xi_n)tanh(y)}, \quad (8)$$

It's similar to Case ii and Case iii. While from Case v to Case viii, it is not so clear to represent F_{n+p_s} by a rational fractal of F_n. So we make a change of the form of F_n. For example, in Case vii, we can rewrite $\sec\xi + \tan\xi$ as $\frac{1+sin(\xi_n)}{cos(\xi_n)}$. Then we can get the expression of U_{n+p_s} in $sin(\xi_n)$ and $cos(\xi_n)$.

Step2: Derive and solve the algebraic system

Table 1: Solution of (4)

case	p	q	r	F
i	1	0	-1	$\tanh\xi, \coth\xi$
ii	1	0	1	$\tan\xi$
iii	-1	0	-1	$\cot\xi$
iv	0	1	-1	$\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2}\xi)$
v	0	-1	1	$\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$
vi	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth\xi \pm \mathrm{csch}\xi, \tanh\xi \pm i\mathrm{sech}\xi$
vii	$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec\xi+\tan\xi, \csc\xi-\cot\xi$
viii	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec\xi-\tan\xi, \csc\xi+\cot\xi$
viiii	0	0	$\neq 0$	$-\frac{1}{r\xi+\lambda}$(λ is arbitrary constant)

Substitute F_n and (3) into (6). Clearring the denominator and collecting the coefficients of F_n(or $sin^k(\xi_n)cos^l(\xi_n)$,$k = 0, 1, \cdots, r; l = 0, 1$)and setting them to zero, we can obtain an nonlinear algebra system with respect to a_i, d, c. Solving the algebraic system, we may find the values of unknowns.

Step 3: Build and test the exact solutions of (1).

Substitute a_i, d, c gotten in Step 3, F_n, (2) and (3). Thus we can obtain the traveling wave solutions of nonlinear DDE (1).

3 New solutions of Toda lattice

The classic Toda lattice (TL) takes the form

$$\frac{\partial^2 u_n}{\partial t^2} = (\frac{\partial u_n}{\partial t} + 1)(u_{n-1} - 2u_n + u_{n+1}) \tag{9}$$

Follow the first step in the above section, by balancing $\frac{\partial^2 u_n}{\partial t^2}$ and $\frac{\partial u_n}{\partial t}u_n$, we get

$$u(t) = a_0 + a_1 F_n(\xi_n) + \frac{a_2}{F_n(\xi_n)}, \tag{10}$$

where $F_n(\xi_n)$ satisfies (4). Next we will give solutions to (1) according to Table 1.

3.1 Case i to Case iii

In Case i, we set

$$u_n(t) = a_0 + a_1 tanh(\xi_n) + \frac{a_2}{tanh(\xi_n)}, \tag{11}$$

Then

$$\begin{aligned} u_{n\pm1}(t) &= a_0 + a_1 tanh(\xi_n \pm d) + \frac{a_2}{tanh(\xi_n \pm d)} \\ &= a_0 + a_1 \frac{tanh(\xi_n) \pm tanh(d)}{1 \pm tanh(\xi_n)tanh(d))} + a_2 \frac{1 \pm tanh(\xi_n)tanh(d)}{tanh(\xi_n) \pm tanh(d)}, \end{aligned} \tag{12}$$

Substituting (11) and (12) in to (9), after clearing the denominator , we get an algebraic system which is omitted because of complexity. With the aid of Maple,we get the solutions for the algebraic system, then we getthe following solutions of (1)

$$u_n(t)_1 = a_0 \pm \frac{1}{2} sinh(d) tanh(\xi_n), \tag{13}$$

$$u_n(t)_2 = a_0 \pm \frac{1}{2} sinh(d) coth(\xi_n), \tag{14}$$

where $\xi_n = dn \pm sinh(d)t + \xi_0$.

$$u_n(t)_3 = a_0 \pm \frac{1}{2} sinh(2d)(tanh(\xi_n) + coth(\xi_n)), \tag{15}$$

where $\xi_n = dn \pm \frac{1}{2} sinh(2d)$

3.2 Case iv

Here we let

$$u_n = a_0 + a_1\left(\frac{1}{2} + \frac{1}{2}tanh\left(\frac{1}{2}\xi_n\right)\right) + a_2\frac{1}{\frac{1}{2} + \frac{1}{2}tanh\left(\frac{1}{2}\xi_n\right)} \tag{16}$$

$$u_{n\pm 1} = a_0 + a_1\left(\frac{1}{2} + \frac{1}{2}tanh\left(\frac{1}{2}\xi_n \pm \frac{d}{2}\right)\right) + a_2\frac{1}{\frac{1}{2} + \frac{1}{2}tanh\left(\frac{1}{2}\xi_n \pm \frac{d}{2}\right)} \tag{17}$$

we use

$$tanh\left(\frac{1}{2}\xi_n \pm \frac{d}{2}\right) = \frac{tanh(\frac{1}{2}\xi) \pm tanh(\frac{d}{2})}{(1 \pm tanh(\frac{1}{2}\xi)tanh(\frac{d}{2}))};$$

to simplify the above two equations. Similar to the above subsection, substituting (16)-(**??**) to (**??**), clearing the dominator, we get an algebraic system . Then we get solutions of the algebraic system and built solution to the original Toda equation(9) as

$$\begin{aligned} u_n(t)_4 &= a_0 \pm 2sinh\left(\frac{d}{2}\right)\left(\frac{1}{2} + \frac{1}{2}tanh\left(\frac{1}{2}\xi_n\right)\right) \\ &= a_0 \pm sinh(\frac{d}{2}) \pm sinh(\frac{d}{2})tanh(\frac{1}{2}dn \pm sinh(\frac{d}{2})t + \xi_0). \end{aligned} \tag{18}$$

We note that this solution is the same to (13). Similarly, by Case v we get the same solution as (14).

3.3 Case vi

Here

$$u_n = a_0 + a_1\left(\frac{1}{2} - \frac{1}{2}coth\left(\frac{1}{2}\xi_n\right)\right) + a_2\frac{1}{\frac{1}{2} - \frac{1}{2}coth\left(\frac{1}{2}\xi_n\right)} \tag{19}$$

$$u_{n\pm 1} = a_0 + a_1\left(\frac{1}{2} - \frac{1}{2}coth\left(\frac{1}{2}\xi_n \pm \frac{d}{2}\right)\right) + a_2\frac{1}{\frac{1}{2} - \frac{1}{2}coth\left(\frac{1}{2}\xi_n \pm \frac{d}{2}\right)} \tag{20}$$

we use

$$coth\left(\frac{1}{2}\xi_n \pm \frac{d}{2}\right) = \frac{coth(\frac{1}{2}\xi) \pm tanh(\frac{d}{2})}{(1 \pm coth(\frac{1}{2}\xi)tanh(\frac{d}{2}))};$$

to simplify the above two equations.

Substituting (19) to (9), simplifying and clearing the dominator, we get an algebraic systems.Then we get solutions

$$\begin{aligned} u_n(t)_5 &= a_0 \mp 2sinh\left(\frac{d}{2}\right)\left(\frac{1}{2} - \frac{1}{2}coth\left(\frac{1}{2}\xi_n\right)\right) \\ &= a_0 \mp sinh(\frac{d}{2}) \pm sinh(\frac{d}{2})coth(\frac{1}{2}dn \pm sinh(\frac{d}{2})t + \xi_0) \end{aligned} \tag{21}$$

3.4 Case vii: sec-tan solutions

We use

$$sec\xi_n + tan\xi_n = \frac{1 + sin\xi_n}{cos\xi_n},$$

then

$$u_n = a_0 + a_1\frac{1 + sin\xi_n}{cos\xi_n} + a_2\frac{cos\xi_n}{1 + sin\xi_n}, \tag{22}$$

$$u_{n\pm 1} = a_0 + a_1\frac{1 + sin(\xi_n \pm d)}{cos(\xi_n \pm d)} + a_2\frac{cos(\xi_n \pm d)}{1 + sin(\xi_n \pm d)}. \tag{23}$$

The solution for DDE is

$$u_n(t)_6 = a_0 \pm \frac{1}{2}sind(tan(\xi_n) + sec(\xi_n) - \frac{1}{tan(\xi_n) + sec(\xi_n)}), \tag{24}$$

where $\xi_n = dn \mp sindt + \xi_0$. It is a singular solution.

3.5 Case viii: fractal solutions

Here

$$F_n(\xi_n) = -\frac{1}{r\xi_n + \lambda} \tag{25}$$

where λ is an arbitrary constant. Substituting (25) into (10) we get

$$F_{n\pm1}(\xi_n) = -\frac{1}{r\xi_n + \lambda \pm rd} = \frac{F_n(\xi_n)}{1 \mp rdF_n(\xi_n)} \tag{26}$$

Therefor we get the solution as

$$u_n(t)_7 = a_0 + \frac{cr}{r\xi_n + \lambda} + \frac{c^2 - d^2}{d^2cr}(r\xi_n + \lambda). \tag{27}$$

We can see it is a new solution. The graph is given in Fig.1(b) for $r = 1, \lambda = 1, d = 8, c = 1$.

4 Conclusion

In order to find new solutions of DDEs, it is important to find new transformation.

We develop the modified $F-$method to construct exact solutions of DDEs. The method is general and can be also apply to other differential-difference equations.

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