ACADIMIC
World Academic Union

ISSN 1479-3889 (print), 1479-3897 (online)
International Journal of Nonlinear Science
Vol. 3 (2007) No. 2, pp. 150-154

The Hausdorff Measure of the Attractor of an Iterated Function System with Parameter

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(Received 24 October 2006, accepted 31 October 2006)

Abstract: Sierpinski carpet is one of the classic fractals with strict self-similar property. In this paper, we will give a Sierpinski carpet with parameter. When parameter $\theta \in (0, \frac{\pi}{3})$, the lower bound estimate for the Hausdorff measure of this set is obtained by constructing a skillful affine mapping. At the same time, the upper bound for the Hausdorff measure is estimated by the covering of k-th basic intervals. When parameter $\theta \in [\frac{\pi}{3}, \pi)$, by a projecting mapping and the covering of k-th basic intervals,we obtain the exact value of the Hausdorff measure of the attractor of the iterated function system with parameter $\theta \in [\frac{\pi}{3}, \pi)$.

Keywords: Hausdorff measure; Sierpinski carpet; parameter; iterated function system

1 Introduction and main theorem

Computing and estimating the dimension and measure of the fractal sets is one of the important problems in fractal geometry[1−3]. Generally speaking, it is computing the Hausdorff dimension and the Hausdorff measure. For a self-similar set satisfying the open set condition, we know that its Hausdorff dimension equals its self-similar dimension, but there are very few results about the Hausdorff measure, except for a few sets like the Cantor set on the line. Recently, some progress in Sierpinski carpet study have been made[4]. The exact value of Hausdorff measure of a Sierpinski carpet was calculated by Zhou and Wu[5]. On the base of [5], the exact values of Hausdorff measures of some generalized Sierpinski carpets are obtained[6],[7]. In this paper, we shall continue the study on the Hausdorff measures of the attractor of an iterated function system with parameter.

Let F_0 be the isosceles triangle ABC in R^2, $AB = AC = 1$ and $\angle BAC = \theta$. Retaining 3 smaller triangles which are similar to F_0 in F_0 such that they located at the 3 corners of F_0 respectively, and their ratios are $\frac{1}{3}$, respectively. At the same time, the interior of the other part is cut out. Let F_1 be the union of the retained 3 smaller isosceles triangles. In each of the 3 isosceles triangles in F_1, we repeat this process for the last time. We obtain 3^2 smaller isosceles triangles and the union of them be denoted by F_2. We can do the above process infinitely, and obtain $F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_k \supset \cdots$. The nonempty set $F = \bigcap\limits_{k=0}^{\infty} F_k$ is called a $(\frac{1}{3}, \theta)$- Sierpinski carpet.(See Fig 1).

The set F is also the attractor in R^2 for the three contracting maps:

$$\begin{aligned} f_1 \colon (x, y) &\mapsto (\frac{x}{3}, \frac{y}{3}), \\ f_2 \colon (x, y) &\mapsto (\frac{x+2}{3}, \frac{y}{3}), \\ f_3 \colon (x, y) &\mapsto (\frac{x + 2\cos\theta}{3}, \frac{y + 2\sin\theta}{3}), \end{aligned}$$

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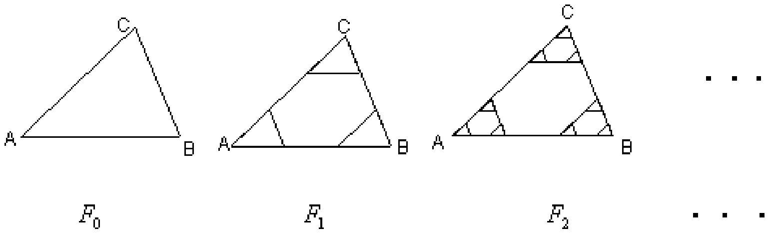

Figure 1: The Generation of the $(\frac{1}{3}, \theta)$-Sierpinski Carpet F

where $\theta \in (0, \pi)$, by which we mean F is the unique nonempty compact set satisfying $f_i(F) \subset F$ for $i = 1, 2, 3$.(See Fig 2).

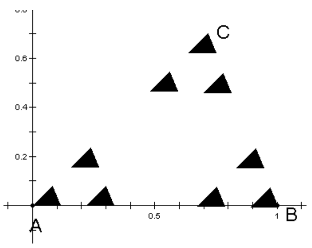

Figure 2: The 2-th Generation of the $(\frac{1}{3}, \theta)$-Sierpinski Carpet F

Since the set F is self-similar and satisfies the open set condition, its Hausdorff dimension is the number s satisfying $3 \cdot (\frac{1}{3})^s = 1$, i.e., $s = 1$. We discuss the Hausdorff measure of the $(\frac{1}{3}, \theta)$- Sierpinski carpet at this dimension.

Theorem 1 *For the Hausdorff dimension $s = 1$, the Hausdorff measure of the $(\frac{1}{3}, \theta)$- Sierpinski carpet F is as follows:*

(i) $\frac{2\sin\theta}{\sqrt{5-4\cos\theta}} \le H^s(F) \le 1$, *with* $0 < \theta < \frac{\pi}{3}$;

(ii) $H^s(F) = 2\sin\frac{\theta}{2}$, *with* $\frac{\pi}{3} \le \theta < \pi$.

2 Some notations and lemmas

Recall that if U is any nonempty subset of n-dimensional space R^n, the diameter of U is defined as $|U| = \sup\{|x - y| : x, y \in U\}$. If $\{U_i\}$ is a countable(or finite) collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F.

Suppose that F is a subset of R^n and s is a non-negative number. For any $\delta > 0$ we define

$$H^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s \quad : \{U_i\} \text{ is a } \delta - \text{cover of } F\}.$$

Thus we look at all covers of F by sets of diameter at most δ and seek to minimize the sum of the s-th powers of the diameters. As δ decreases, the class of permissible covers of F in the above equation is reduced. Therefore, the infimum $H^s_\delta(F)$ increases, and so approaches a limit as $\delta \to 0$. We write

$$H^s(F) = \lim_{\delta \to 0} H^s_\delta(F).$$

This limit exists for any subsets F of R^n, though the limiting value can be (and usually is) 0 or ∞. We call $H^s(F)$ the s-dimensional Hausdorff measure of F. We give the definition of Hausdorff dimension of F as follows:

$$\dim_H F = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

Let D be a closed subset of R^n. A mapping $S : D \to D$ is called a contraction on D if there is a number c with $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in D$. Clearly any contraction is a continuous mapping. If equality holds, i.e. if $|S(x) - S(y)| = c|x - y|$, S transforms sets into geometrically similar ones, and we call S a similarity, c is called the ratio of S.

If there exists a nonempty bounded open set V for any $i \neq j$, $S_i(V) \cap S_j(V) = \emptyset$, such that $\bigcup\limits_{i=0}^{m} S_i(V) \subset V$ with the union disjoint, we say that the S_i satisfy the *open set condition*[8].

Lemma 1 [8] *Let* $S_i \ \ (1 \leq i \leq m)$ *be contractions on* $D \subset R^n$ *so that*

$$|S_i(x) - S_i(y)| \leq c_i|x - y| \ \ (x, y \in D)$$

with $0 < c_i < 1$ *for each* i. *Then there exists a unique nonempty compact set* F *that is invariant for the* S_i, *i.e. which satisfies*

$$F = \bigcup_{i=0}^{m} S_i(F).$$

Lemma 2 [8] *Suppose that the open set condition holds for the similarities* S_i *on* R^n *with ratios* $c_i (1 \leq i \leq m)$. *If* F *is the invariant set satisfying*

$$F = \bigcup_{i=0}^{m} S_i(F),$$

$\dim_H F = \dim_B F = s$, *where* s *is given by* $\sum\limits_{i=1}^{m} c_i^s = 1$. *Moreover, for this value of* s, $0 < H^s(F) < \infty$.

Lemma 3 [8] *Let* $F \subset R^n$ *and* $f : F \to R^n$ *be a mapping such that*

$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in F)$$

for constants $c > 0$. *Then for each* s

$$H^s(f(F)) \leq c^s H^s(F).$$

From Lemma 3. we can get

Lemma 4 *Let* $F \subset R^2$, *we denote orthogonal projection onto* x*-axis by* $proj$, *so that if* F *is a subset of* R^2, *then* $proj(F)$ *is the projection of* F *onto* x*-axis. Clearly,* $|projx - projy| \leq |x - y|$, *i.e.,* $proj$ *is a Lipschitz mapping. Thus, we have*

$$H^s(projF) \leq H^s(F).$$

Lemma 5 [8] *Suppose* F *is a Borel subset of* R^n, *then*

$$H^n(F) = c_n vol^n(F)$$

where the constant $c_n = 2^n \frac{(\frac{n}{2})!}{\pi^{\frac{n}{2}}}$ *is the reciprocal of the volume of an* n*-dimensional ball of diameter 1, vol stands for Lebesgue measure.*

3 The proof of Theorem

From the generation of the $(\frac{1}{3}, \theta)$- Sierpinski carpet F, we can see that for each $k \geq 0$, F_k consists of 3^k isosceles triangles, which were denoted by $\triangle_1^k, \triangle_2^k, \cdots, \triangle_{3^k}^k$. Each $\triangle_i^k$ is called a k-th basic triangle.

Proof of Theorem (i)

It is clear that the 3^k k-th basic triangles of F_k, $\triangle_1^k, \triangle_2^k, \cdots, \triangle_{3^k}^k$ is a covering of F. Let $|\triangle_i^k|$ be the diameter of $\triangle_i^k$, and then through the structure of F and $\theta \in (0, \frac{\pi}{3})$, we have $|\triangle_i^k| = 3^{-k}$. Then by the definition of $H^s(F)$, we can get $H^s_{3^{-k}}(F) \leq \sum\limits_{i=1}^{3^k} |\triangle_i^k| = 3^k \cdot 3^{-k} = 1$ where $s = 1$. Letting $k \to \infty$, then $H^s(F) \leq 1$.

To estimate the lower bound of $H^s(F)$, we let Line-CD be a line that through the points $C(\cos\theta, \sin\theta)$, $D(\frac{1}{2}, 0)$, and construct a vertical line of Line-CD that through the point $A(0,0)$, which we denoted by Line-AH. We denote orthogonal projection onto Line-AH by f, and then $f(F)$ is the projection of F onto Line-AH. (See Fig 3). It is easy to see that Line-CD parallel with Line-A_2B_1.We denote $f(F)$ is the projection onto Line-AH of F. Obviously, $f(F)$ is the line segment AH.

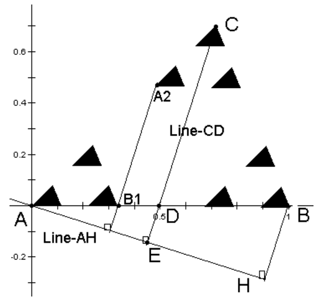

Figure 3: Projection of the Sierpinski Carpet F on the Line-AH

It is easy to see that f is a Lipschitz mapping. Thus, by Lemma 3 and Lemma 4, we have $H^s(f(F)) \leq H^s(F)$. As we know, $f(F)$ is the line segment AH. By Lemma 5, we have

$$H^n(f(F)) = c_n^{-1}\mathrm{vol}^n F = |f(F)| = |AH|,$$

where $n = 1$. By computing, we have $|AH| = \frac{2\sin\theta}{\sqrt{5-4\cos\theta}}$. Therefore, we have $H^s(F) \geq H^s(f(F)) = \frac{2\sin\theta}{\sqrt{5-4\cos\theta}}$, where $\theta \in (0, \frac{\pi}{3})$.

Proof of Theorem (ii)

It is clear that the 3^k k-th basic triangles of F_k, $\triangle_1^k, \triangle_2^k, \cdots, \triangle_{3^k}^k$ is a covering of F. From the structure of F and $\theta \in [\frac{\pi}{3}, \pi)$, and the fundamental property of the triangles, we have $|\triangle_i^k| = 3^{-k}2\sin\frac{\theta}{2}$.

Thus $H^s_{3^{-k}2\sin\frac{\theta}{2}}(F) \leq \sum\limits_{i=1}^{3^k} |U_i| = 3^k \cdot |3^{-k}2\sin\frac{\theta}{2}|^s = 2\sin\frac{\theta}{2}$ with $s = 1$. Letting $k \to \infty$, then $H^s(F) \leq 2\sin\frac{\theta}{2}$.

Let us see the graph in Fig 4, which the vertex B of triangle locates at O, and the side BC lies in x-axis.

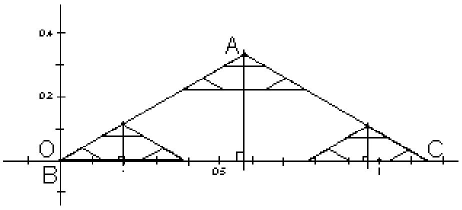

Figure 4: Projection of the Sierpinski Carpet F on the horizontal axis

Now, we denote orthogonal projection onto x-axis by proj, so that projF is the projection of F onto x-axis. Clearly, proj is a Lipschitz mapping. Thus, by Lemma 3 and Lemma 4 , we have $H^s(\text{projF}) \leq H^s(F)$. As a sequence ,we need to compute the value of $H^s(\text{projF})$. It is easy to see that projF is the line segment BC on the x-axis. Therefore,by lemma 5, we have

$$H^n(\text{projF}) = c_n^{-1}\mathrm{vol}^n F = |BC| = 2\sin\frac{\theta}{2},$$

where $n = 1$. We have $H^s(F) \geq H^s(\text{projF}) = 2\sin\frac{\theta}{2}$, with $s = 1$, where $\theta \in [\frac{\pi}{3}, \pi)$.

4 Conclusion

In this paper, we use the projection to calculate the lower bound of the Hausdorff measure of the $(\frac{1}{3},\theta)$-Sierpinski carpets, which is simpler than using the mass distribution. And the exact values of Hausdorff measure of a class of called $(\frac{1}{3},\theta)$-Sierpinski carpets with parameter $\theta \in [\frac{\pi}{3},\pi)$ which Hausdorff dimension equals 1, are obtained. The Hausdorff measure of some classic Sierpinski carpets can be obtained with this method.

Acknowledgements

Research is supported by the National Science Foundation of China (10671180) and Jiangsu University 05JDG041.

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