The Hausdorff Measure of the Attractor of an Iterated Function System with Parameter

Dehua Liu, Meifeng Dai *
Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University
Zhenjiang, Jiangsu, 212013, China
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Abstract: Sierpinski carpet is one of the classic fractals with strict self-similar property. In this paper, we will give a Sierpinski carpet with parameter. When parameter \( \theta \in (0, \frac{\pi}{3}) \), the lower bound estimate for the Hausdorff measure of this set is obtained by constructing a skillful affine mapping. At the same time, the upper bound for the Hausdorff measure is estimated by the covering of \( k \)-th basic intervals. When parameter \( \theta \in \left[ \frac{\pi}{3}, \pi \right) \), by a projecting mapping and the covering of \( k \)-th basic intervals, we obtain the exact value of the Hausdorff measure of the attractor of the iterated function system with parameter \( \theta \in \left[ \frac{\pi}{3}, \pi \right) \).

Keywords: Hausdorff measure; Sierpinski carpet; parameter; iterated function system

1 Introduction and main theorem

Computing and estimating the dimension and measure of the fractal sets is one of the important problems in fractal geometry\(^1\,\text{-}\,3\). Generally speaking, it is computing the Hausdorff dimension and the Hausdorff measure. For a self-similar set satisfying the open set condition, we know that its Hausdorff dimension equals its self-similar dimension, but there are very few results about the Hausdorff measure, except for a few sets like the Cantor set on the line. Recently, some progress in Sierpinski carpet study have been made\(^4\). The exact value of Hausdorff measure of a Sierpinski carpet was calculated by Zhou and Wu\(^5\). On the base of \(^5\), the exact values of Hausdorff measures of some generalized Sierpinski carpets are obtained\(^6\,\text{-}\,7\). In this paper, we shall continue the study on the Hausdorff measures of the attractor of an iterated function system with parameter.

Let \( F_0 \) be the isosceles triangle \( ABC \) in \( \mathbb{R}^2 \), \( AB = AC = 1 \) and \( \angle BAC = \theta \). Retaining 3 smaller triangles which are similar to \( F_0 \) in \( F_0 \) such that they located at the 3 corners of \( F_0 \) respectively, and their ratios are \( \frac{1}{3} \), respectively. At the same time, the interior of the other part is cut out. Let \( F_1 \) be the union of the retained 3 smaller isosceles triangles. In each of the 3 isosceles triangles in \( F_1 \), we repeat this process for the last time. We obtain \( 3^2 \) smaller isosceles triangles and the union of them be denoted by \( F_2 \). We can do the above process infinitely, and obtain \( F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_k \supset \cdots \). The nonempty set \( F = \bigcap_{k=0}^{\infty} F_k \) is called a \( (\frac{1}{3}, \theta) \)-Sierpinski carpet.(See Fig 1).

The set \( F \) is also the attractor in \( \mathbb{R}^2 \) for the three contracting maps:

\[
\begin{align*}
  f_1: (x, y) &\mapsto \left( \frac{x}{3}, \frac{y}{3} \right), \\
  f_2: (x, y) &\mapsto \left( \frac{x + 2}{3}, \frac{y}{3} \right), \\
  f_3: (x, y) &\mapsto \left( \frac{x + 2 \cos \theta}{3}, \frac{y + 2 \sin \theta}{3} \right),
\end{align*}
\]

*Corresponding author. E-mail address: daimf@ujs.edu.cn

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Figure 1: The Generation of the \((\frac{1}{3}, \theta)\)-Sierpinski Carpet F

where \(\theta \in (0, \pi)\), by which we mean \(F\) is the unique nonempty compact set satisfying \(f_i(F) \subset F\) for \(i = 1, 2, 3\).

Figure 2: The 2-th Generation of the \((\frac{1}{3}, \theta)\)-Sierpinski Carpet F

Since the set \(F\) is self-similar and satisfies the open set condition, its Hausdorff dimension is the number \(s\) satisfying \(3 \cdot (\frac{1}{3})^s = 1\), i.e., \(s = 1\). We discuss the Hausdorff measure of the \((\frac{1}{3}, \theta)\)-Sierpinski carpet at this dimension.

**Theorem 1** For the Hausdorff dimension \(s = 1\), the Hausdorff measure of the \((\frac{1}{3}, \theta)\)-Sierpinski carpet \(F\) is as follows:

(i) \(\frac{2\sin \theta}{\sqrt{5 - 4 \cos \theta}} \leq H^s(F) \leq 1\), with \(0 < \theta < \frac{\pi}{3}\); 
(ii) \(H^s(F) = 2 \sin \frac{\theta}{2}\), with \(\frac{\pi}{3} \leq \theta < \pi\).

2 Some notations and lemmas

Recall that if \(U\) is any nonempty subset of \(n\)-dimensional space \(R^n\), the diameter of \(U\) is defined as \(|U| = \sup \{|x - y| : x, y \in U\}\). If \(\{U_i\}\) is a countable(or finite) collection of sets of diameter at most \(\delta\) that cover \(F\), i.e. \(F \subset \bigcup_{i=1}^{\infty} U_i\) with \(0 < |U_i| \leq \delta\) for each \(i\), we say that \(\{U_i\}\) is a \(\delta\)-cover of \(F\).

Suppose that \(F\) is a subset of \(R^n\) and \(s\) is a non-negative number. For any \(\delta > 0\) we define

\[
H^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta - \text{cover of } F \right\}.
\]

Thus we look at all covers of \(F\) by sets of diameter at most \(\delta\) and seek to minimize the sum of the \(s\)-th powers of the diameters. As \(\delta\) decreases, the class of permissible covers of \(F\) in the above equation is reduced. Therefore, the infimum \(H^s_\delta(F)\) increases, and so approaches a limit as \(\delta \to 0\). We write

\[
H^s(F) = \lim_{\delta \to 0} H^s_\delta(F).
\]

This limit exists for any subsets \(F\) of \(R^n\), though the limiting value can be (and usually is) 0 or \(\infty\). We call \(H^s(F)\) the \(s\)-dimensional Hausdorff measure of \(F\). We give the definition of Hausdorff dimension of \(F\) as follows:

\[
\dim_H F = \inf \{s : H^s(F) = 0\} = \sup \{s : H^s(F) = \infty\}.
\]

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Let \( D \) be a closed subset of \( R^n \). A mapping \( S : D \rightarrow D \) is called a contraction on \( D \) if there is a number \( c \) with \( 0 < c < 1 \) such that \( |S(x) - S(y)| \leq c|x - y| \) for all \( x, y \in D \). Clearly any contraction is a continuous mapping. If equality holds, i.e. if \( |S(x) - S(y)| = c|x - y| \), \( S \) transforms sets into geometrically similar ones, and we call \( S \) a similarity, \( c \) is called the ratio of \( S \).

If there exists a nonempty bounded open set \( V \) for any \( i \neq j \), \( S_i(V) \cap S_j(V) = \emptyset \), such that \( \bigcup_{i=0}^{m} S_i(V) \subset V \) with the union disjoint, we say that the \( S_i \) satisfy the open set condition\(^8\).

**Lemma 1**\(^8\) Let \( S_i \ (1 \leq i \leq m) \) be contractions on \( D \subset R^n \) so that
\[
|S_i(x) - S_i(y)| \leq c_i|x - y| \quad (x, y \in D)
\]
with \( 0 < c_i < 1 \) for each \( i \). Then there exists a unique nonempty compact set \( F \) that is invariant for the \( S_i \), i.e. which satisfies
\[
F = \bigcup_{i=0}^{m} S_i(F).
\]

**Lemma 2**\(^8\) Suppose that the open set condition holds for the similarities \( S_i \) on \( R^n \) with ratios \( c_i(1 \leq i \leq m) \). If \( F \) is the invariant set satisfying
\[
F = \bigcup_{i=0}^{m} S_i(F),
\]
\( \dim_H F = \dim_B F = s \), where \( s \) is given by \( \sum_{i=1}^{m} c_i^s = 1 \). Moreover, for this value of \( s \), \( 0 < H^s(F) < \infty \).

**Lemma 3**\(^8\) Let \( F \subset R^n \) and \( f : F \rightarrow R^n \) be a mapping such that
\[
|f(x) - f(y)| \leq c|x - y| \quad (x, y \in F)
\]
for constants \( c > 0 \). Then for each \( s \)
\[
H^s(f(F)) \leq c^s H^s(F).
\]

From Lemma 3. we can get

**Lemma 4** Let \( F \subset R^2 \), we denote orthogonal projection onto \( x \)-axis by \( \text{proj} \), so that if \( F \) is a subset of \( R^2 \), then \( \text{proj}(F) \) is the projection of \( F \) onto \( x \)-axis. Clearly, \( |\text{proj}x - \text{proj}y| \leq |x - y| \), i.e., \( \text{proj} \) is a Lipschitz mapping. Thus, we have
\[
H^s(\text{proj}F) \leq H^s(F).
\]

**Lemma 5**\(^8\) Suppose \( F \) is a Borel subset of \( R^n \), then
\[
H^n(F) = c_n \text{vol}^n(F)
\]
where the constant \( c_n = 2^n \left( \frac{2}{\pi} \right)^{n/2} \) is the reciprocal of the volume of an \( n \)-dimensional ball of diameter 1, \( \text{vol} \) stands for Lebesgue measure.

## 3 The proof of Theorem

From the generation of the \((\frac{1}{3}, \theta)\)-Sierpinski carpet \( F \), we can see that for each \( k \geq 0 \), \( F_k \) consists of \( 3^k \) isosceles triangles, which were denoted by \( \Delta_{1}^{k}, \Delta_{2}^{k}, \ldots, \Delta_{3^k}^{k} \). Each \( \Delta_{i}^{k} \) is called a \( k \)-th basic triangle.

**Proof of Theorem (i)**

It is clear that the \( 3^k \) \( k \)-th basic triangles of \( F_k \), \( \Delta_{1}^{k}, \Delta_{2}^{k}, \ldots, \Delta_{3^k}^{k} \) is a covering of \( F \). Let \( |\Delta_{i}^{k}| \) be the diameter of \( \Delta_{i}^{k} \), and then through the structure of \( F \) and \( \theta \in (0, \frac{\pi}{3}) \), we have \( |\Delta_{i}^{k}| = 3^{-k} \). Then by the
definition of $H^s(F)$, we can get $H^s_{3-k}(F) \leq \sum_{i=1}^{2^k} |\triangle_i^k| = 3^k \cdot 3^{-k} = 1$ where $s = 1$. Letting $k \to \infty$, then $H^s(F) \leq 1$.

To estimate the lower bound of $H^s(F)$, we let Line-CD be a line that through the points $C(\cos \theta, \sin \theta)$, $D(\frac{1}{2}, 0)$, and construct a vertical line of Line-CD that through the point $A(0, 0)$, which we denoted by Line-AH. We denote orthogonal projection onto Line-AH by $f$, and then $f(F)$ is the projection of $F$ onto Line-AH. (See Fig 3). It is easy to see that Line-CD parallel with Line-$A_2B_1$. We denote $f(F)$ is the projection onto Line-AH of $F$. Obviously, $f(F)$ is the line segment $AH$.

\[ \text{Figure 3: Projection of the Sierpinski Carpet } F \text{ on the Line-AH} \]

It is easy to see that $f$ is a Lipschitz mapping. Thus, by Lemma 3 and Lemma 4, we have $H^s(f(F)) \leq H^s(F)$. As we know, $f(F)$ is the line segment $AH$. By Lemma 5, we have

\[ H^n(f(F)) = c_n^{-1} \text{vol}^n F = |f(F)| = |AH|, \]

where $n = 1$. By computing, we have $|AH| = \frac{2 \sin \theta}{\sqrt{3 - 4 \cos \theta}}$. Therefore, we have $H^s(F) \geq H^s(f(F)) = \frac{2 \sin \theta}{\sqrt{3 - 4 \cos \theta}}$, where $\theta \in (0, \frac{\pi}{3})$.

**Proof of Theorem (ii)**

It is clear that the $3^k$ $k$-th basic triangles of $F_k$, $\triangle_1^k, \triangle_2^k, \cdots, \triangle_{3^k}^k$ is a covering of $F$. From the structure of $F$ and $\theta \in [\frac{\pi}{2}, \pi)$, and the fundamental property of the triangles, we have $|\triangle_i^k| = 3^{-k}2 \sin \frac{\theta}{2}$.

Thus $H^s_{3^{-k}2 \sin \frac{\theta}{2}}(F) \leq \sum_{i=1}^{3^k} |U_i| = 3^k \cdot 3^{-k}2 \sin \frac{\theta}{2} = 2 \sin \frac{\theta}{2}$ with $s = 1$. Letting $k \to \infty$, then $H^s(F) \leq 2 \sin \frac{\theta}{2}$.

Let us see the graph in Fig 4, which the vertex $B$ of triangle locates at $O$, and the side $BC$ lies in $x$-axis.

\[ \text{Figure 4: Projection of the Sierpinski Carpet } F \text{ on the horizontal axis} \]

Now, we denote orthogonal projection onto $x$-axis by proj, so that $\text{proj}F$ is the projection of $F$ onto $x$-axis. Clearly, proj is a Lipschitz mapping. Thus, by Lemma 3 and Lemma 4, we have $H^s(\text{proj}F) \leq H^s(F)$. As a sequence, we need to compute the value of $H^s(\text{proj}F)$. It is easy to see that $\text{proj}F$ is the line segment $BC$ on the $x$-axis. Therefore, by Lemma 5, we have

\[ H^n(\text{proj}F) = c_n^{-1} \text{vol}^n F = |BC| = 2 \sin \frac{\theta}{2}, \]

where $n = 1$. We have $H^s(F) \geq H^s(\text{proj}F) = 2 \sin \frac{\theta}{2}$, with $s = 1$, where $\theta \in [\frac{\pi}{2}, \pi)$.
4 Conclusion

In this paper, we use the projection to calculate the lower bound of the Hausdorff measure of the \((\frac{1}{3}, \theta)\)-Sierpinski carpets, which is simpler than using the mass distribution. And the exact values of Hausdorff measure of a class of called \((\frac{1}{3}, \theta)\)-Sierpinski carpets with parameter \(\theta \in \left[\frac{\pi}{3}, \pi\right)\) which Hausdorff dimension equals 1, are obtained. The Hausdorff measure of some classic Sierpinski carpets can be obtained with this method.

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References


