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The Dynamics of Triopoly Game with Heterogeneous Players

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Abstract: In this paper, the dynamics of discrete triopoly game with heterogeneous players has been studied. In this game, three players with different expectations were used to iterate the dynamic game. The three players were considered to be: boundedly rational, adaptive and naive. The investigation of the fixed points and their stability, bifurcation diagrams, strange attractors and chaotic behavior were analyzed. Also Lyapunov exponents and fractal dimension of chaotic attractor of our map, which is equivalent to dimension of He/non like map was calculated. It has been deduced that, the heterogeneous belief of players can lead to instability, rich dynamics and complexity.

Key words: Triopoly game, Chaotic behavior, Heterogeneous expectations, Fractal dimension.

1 Introduction

The classical oligopoly model was proposed in (1883) by the French economist Augustine Cournot [1]. Oligopoly is a market system of competitive players which may produce homogeneous output to be sold a common market. The dynamical oligopoly game is a complex, because oligopolist must consider not only behaviors of the consumers, but also those of the competitors and their reactions. In oligopoly model all players are maximize their profits. While, triopoly game is oligopoly market with three players. Recently, the dynamics of triopoly game have been studied in [2,3,4,5]. Triopoly game with homogeneous players has been studied by Puu [2]. He considered that all players in this game are naive player and showed that the dynamics of triopoly game can lead to complex dynamics such as cycles and chaos. Agiza et al. [3] extended the Cournot duopoly game to triopoly case and studied the multistability of game. Discrete time dynamical triopoly games with homogeneous expectations have been found to be of economically interest [2,3,4,5]. In these studies the players were considered to have the same expectations rules for compute expected outputs. In addition, they showed that the dynamical Cournot oligopoly game may have times which never settle to a steady state, and in the long run the game exhibits bounded dynamics which may be periodic, or quasi-periodic or chaotic.

Expectations play a key role in modeling dynamic phenomena in economics and finance. A producer (player) can choose his own way out of many available techniques to adjustment production capacity and many different types of behaviors coexist in reality. In the market of triopoly game each player may behaves differently expectations strategies. Later economic models with heterogeneous agents is introduced [6,7]. Heterogeneous expectations, dynamics and stability of market have been reviewed in [8]. Dynamics of heterogeneous two-dimensional cobweb model have been examined by Onozaki et al. in [9]. Recently, Agiza and Elsadany [10,11] examined the dynamics of Cournot duopoly game with heterogeneous players.

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In the present work, we consider the triopoly game which player takes different strategy for to compute his expected output. Such choice makes the triopoly game with heterogeneous players are more reality than homogeneous case. In addition, this study provides an example in which a slight behavioral heterogeneity could be factor generates complex dynamics of triopoly game with linear demand function and linear cost function. We take player 1 represents a boundedly rational player [12] while player 2 is adaptive and player 3 is naive.

The paper is organized as follows, the time evaluation of the dynamical triopoly game with heterogeneous players is described in section 2. In section 3, we determine the existence and local stability of fixed points of invertible three dimensional map. Chaotic behavior under some change of control parameters of the game is investigated by numerical simulation in section 4. Also Lyapunov exponent and fractal dimension of the strange attractor of our map is measured numerically. Finally, section 5 concludes of this paper.

2 Heterogeneous players

In oligopoly game, players can choose simple expectations rules such as naive or complicated as adaptive and bounded rationality. The players can use the same strategy (homogeneous expectations) or can use different strategy (heterogeneous expectations). In this section we shall deduce the framework equations of triopoly game with fully heterogeneous players. In the game of fully heterogeneous players each player think with different strategy to maximize his output. So, in triopoly game we propose three different players are: boundedly rational player, adaptive player and naive player.

We consider Cournot triopoly game where q_i denotes the quantity supplied by ith firm, $i = 1$, 2, 3. In addition let $P(q_i + q_j)$, $i \neq j$, denotes a twice differentiable and nonincreasing inverse demand function and let $C_i(q_i)$ denote the twice differentiable increasing cost function. For the ith firm the profit function resulting is given by

$$\Pi_i(q_i, q_j) = P(q_i + q_j) - C_i(q_i), \quad i = 1,\ 2,\ 3. \tag{1}$$

We Consider a simple Cournot type triopoly market where players produce the same goods or homogeneous goods which are prefect substitutes and over them at discrete-time periods $t = 0,\ 1,\ 2, ...$ on a common market. At each period t, every player must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period $t + 1$. If we denote by $q_i(t)$ the output of ith player at time period time t, then its production $q_i(t + 1)$ for the next time period $t + 1$ is decided by solving the optimization problem

$$q_i = r_i(q_j) = \arg\max_{q_i} [P(\ q_i + q_j) - C_i(q_i)]. \tag{2}$$

In our game, we assume that first player is boundedly rational player, hence it does not has a complete knowledge of the demand function of the market, and builds his output decision on the basis of the expected marginal profit $\frac{\partial \Pi_1}{\partial q_1}$ [12]. If the marginal profit is positive (negative) he increase (decreases) its production q_i at next period output. Then the dynamical equation of player 1 has the form

$$q_1(t+1) = q_1(t) + \alpha q_1(t)\frac{\partial \Pi_1}{\partial q_1},\ t = 0, 1, 2, ... \tag{3}$$

where α is a positive parameter which represents the relative speed of adjustment of boudedly rational player. We assume player 2 is adaptive player. When player 2 think with adaptive expectation hence he compute his output with weight between last periods' output q_2 and his reaction function $r_2(q_1, q_3)$. By using above assumptions and using Eq. (2), hence the dynamical equation of the adaptive player 2 is given by

$$q_2(t+1) = (1-\beta)q_2(t) + \beta r_2(q_1(t), q_3(t)), \tag{4}$$

where $\beta \in [0, 1]$ is speed of adjustment of adaptive player.

When the player 3 is naive player he compute his outputs from using the reaction function in equation (2). If we assume that the optimization problem in Eq.(2) have unique solutions, so the dynamical equation for player 3 has the form

$$q_3(t+1) = r_3(q_1(t) + q_2(t)), \tag{5}$$

where r_3 is often referred to as Best replies (or reaction function of player 3).

Therefore, the dynamical triopoly game in this case is formed from combining Eqs. (3-5), and has the form

$$\begin{cases} q_1(t+1) = q_1(t) + \alpha q_1(t)\frac{\partial \Pi_1}{\partial q_1} \\ q_2(t+1) = (1-\beta)q_2(t) + \beta r_2(q_1(t), q_3(t)) \\ q_3(t+1) = r_3(q_1(t) + q_2(t)) \end{cases} \tag{6}$$

Hence, the system (6) describes the dynamical triopoly game with fully heterogeneous players (three different players). In the next section, we are study the dynamical behaviors of this model with linear demand function and linear cost function.

3 Triopoly game with heterogeneous players

We assume that there are three players producing the homogeneous goods in oligopoly market. Let $q_i(t),\ i = 1, 2, 3$ represent the output of ith player during period $t,$ with the production cost function $C_i(q_i).$ The price prevailing in period $t,$ is determine by the total supply $Q(t) = q_1(t) + q_2(t) + q_3(t)$ through the linear demand function

$$P = f(Q) = a - bQ, \tag{7}$$

where $a, b > 0$. We take the cost function in the linear form

$$C_i(q_i) = c_i q_i, \quad i = 1, 2, 3 \tag{8}$$

where c_i is the marginal cost of ith player. With these assumptions the single profit of ith player is given

$$\begin{aligned} \Pi_i(q_1, q_2, q_3) &= Pq_i - C_i(q_i) \\ &= q_i(a - bQ) - c_i q_i, \quad i = 1, 2, 3. \end{aligned} \tag{9}$$

Then the marginal profit of ith player at the point (q_1, q_2, q_3) is given by

$$\frac{\partial \Pi_i}{\partial q_i} = a - c_i - 2bq_i - bq_j, \quad i, j = 1, 2, 3,\ j \neq i, \tag{10}$$

this optimization problem has unique solution in the form

$$q_i = \frac{1}{2b}(a - c_i - bq_j). \tag{11}$$

In this study, we consider three different players expectations are proposed; boundedly rational player, adaptive player and naive player. By inserting Eqs. (10) and (11) into (6), we have the triopoly game with heterogeneous players in the form

$$\begin{cases} q_1(t+1) = q_1(t) + \alpha q_1(t)(a - 2bq_1(t) - b(q_2(t) + q_3(t)) - c_1) \\ q_2(t+1) = (1-\beta)q_2(t) + \frac{\beta}{2b}(a - b(q_1(t) + q_3(t)) - c_2) \\ q_3(t+1) = \frac{1}{2b}(a - b(q_1(t) + q_2(t)) - c_3) \end{cases}. \tag{12}$$

System (12) is a three-dimensional invertible map which depends on six parameters. We are concerned with the qualitative changes of the asymptotic and/or long-run dynamics of the iterated map (12). We will study the asymptotic behavior of the dynamical model using the localization of fixed points of the dynamical system and the determination of the parameters sets for given local stable fixed points. The fixed points of the map (12), are the solution of the following algebraic system

$$\begin{aligned} & q_1(a - 2bq_1 - b(q_2 + q_3) - c_1) = 0 \\ & (a - 2bq_2 - b(q_1 + q_3) - c_2) = 0 \\ & (a - 2bq_3 - b(q_1 + q_2) - c_3) = 0 \end{aligned}. \tag{13}$$

Note that the algebraic system (13) does not depend on the adjustment of boundedly rational player α. From system (13) one can get the following two fixed points:

$$E_1 = (0, \tfrac{a-2c_2+c_3}{3b}, \tfrac{a-2c_3+c_2}{3b}),$$
$$E_* = (q_1^*, q_2^*, q_3^*) = (\tfrac{a-3c_1+c_2+c_3}{4b}, \tfrac{a-3c_2+c_1+c_3}{4b}, \tfrac{a-3c_3+c_1+c_2}{4b})$$

and provided that by following conditions:

$$\begin{cases} 3c_1 - c_2 - c_3 < a \\ 3c_2 - c_1 - c_3 < a \\ 3c_3 - c_1 - c_2 < a \end{cases}. \tag{14}$$

where E_1 is the boundary equilibrium and E_* is the Nash equilibrium point.

Now, we shall discuss the dynamical behavior of system (12) around E_1 and E_*. To investigate the stability of each fixed point, we return to the three dimensional map (12). Let J be the Jacobian matrix of the system (12) corresponding to the state variables (q_1, q_2, q_3) then

$$J(q_1,q_2,q_3) = \begin{pmatrix} 1+\alpha(a-4bq_1-b(q_2+q_3)-c_1) & -\alpha bq_1 & -\alpha bq_1 \\ -\frac{\beta}{2} & 1-\beta & -\frac{\beta}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}. \tag{15}$$

Proposition 1 The boundary equilibrium point E_1 of the dynamical system (12) is a saddle point.

Proof: In order to prove this results, we consider the eigenvalues of Jacobian matrix J at E_1 which take the form

$$J(E_1) = \begin{pmatrix} 1+\frac{\alpha(a+c_2+c_3-3c_1)}{3} & 0 & 0 \\ -\frac{\beta}{2} & 1-\beta & -\frac{\beta}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},$$

and , consequently, the eigenvalues are $\lambda_1 = 1+\frac{\alpha(a+c_2+c_3-3c_1)}{3}$ and $\lambda_{2,3} = \frac{1}{2}-\frac{\beta}{2}\pm\frac{1}{2}\sqrt{1-\beta+\beta^2}$. It is clear that when conditions Eq. (14) hold, then $|\lambda_1| > 1$ and $|\lambda_{2,3}| < 1$. Therefore, E_1 is saddle point of the game (12).

3.1 Local stability of Nash equilibrium point

Now we consider the stability properties of E_*. Linearizing the system about Nash equilibrium point E_*, the Jacobian matrix at E_* takes the form

$$J(E_*) = \begin{pmatrix} 1-2\alpha bq_1^* & -bq_1^* & -bq_1^* \\ -\frac{\beta}{2} & 1-\beta & -\frac{\beta}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix},$$

The necessary and sufficient conditions for E_* to be asymptotically stable is all roots of the characteristic equation

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{16}$$

have magnitudes of eigenvalues less than one. Here $A_1 = -2+2\alpha bq_1^*+\beta$, $A_2 = 1-\frac{5}{4}\beta-\frac{5}{2}\alpha bq_1^*+\frac{3}{2}\alpha b\beta q_1^*$ and $A_3 = \frac{1}{4}\beta+\frac{1}{2}\alpha bq_1^*-\frac{1}{2}\alpha b\beta q_1^*$. The determinant of the matrix E_* is less than one, therefore the system is dissipative around Nash equilibrium point. Following the Routh-Hurwitz criterion [13], the roots of Eq. (16) must satisfy $|\lambda| < 1$ for all eigenvalues of corresponding Jacobian matrix $J(E_*)$ if and only if

$$\begin{aligned} &3+A_1-A_2-3A_3 > 0, \\ &1-A_2+A_3(A_1-A_3) > 0, \\ &1-A_1+A_2-A_3 > 0. \end{aligned} \tag{17}$$

According to Routh-Hurwitz criteria (17), The Nash equilibrium point E_* is locally asymptotically stable if

$$12\beta-3\beta^2-(24\alpha b+4\alpha\beta b-4\alpha b\beta^2)q_1^*+(8\alpha^2b^2\beta+4\alpha^2\beta^2b^2-12\alpha^2b^2)q_1^{*^2} < 0$$

and

$$-8+5\beta+10\alpha bq_1^*-4\alpha\beta bq_1^*<0$$

It is obvious from the numerical simulation that an increase of the speed of adjustment of boundedly rational player, holding the other parameters fixed, has a destabilizing effect. In fact, an increase of α starting from a set of parameters which ensures the local stability of Nash equilibrium, can bring the point (α,β) out the region of stability, through the flip bifurcation curve. In this case the region of stability becomes small, and this can cause a loss of stability of E_*. The other parameters of the system have similar arguments. From the above, one can conclude that the stability of E_* depends on the system parameters. When the parameter α increases, complex dynamics such as period doubling and strange attractors are generated where the maximum Lyapunov exponents of the system (12) become positive see numerical simulations.

4 Numerical simulations

In this section we study the dynamical behaviors of triopoly game with heterogeneous players to show that the behavior is of more complicated dynamics features than the case of homogeneous players. To provide some numerical evidence for the chaotic behavior of the dynamical game (12), we will present the various numerical results, such as bifurcation diagrams, strange attractors, Lyapunov exponents, sensitive dependence on initial conditions and fractal dimension, to show the included chaotic behavior.

In order to study the local stability properties of the Nash equilibrium point, it is convenient to consider the following set of parameters a = 10,b = .5,c_1 = 1,c_2 = 2 , c_3 = 3 and $\beta = 0.33$. Figure 1 shows the bifurcation diagram with respect to α (speed of adjustment of boundedly rational player), and maximal Lyapunov exponent with respect to α of the dynamical game (12). In this figure, the positive values show that the solutions have chaotic behaviors. The bifurcation scenario is occurred, if α is small consequently, there is a stable Nash equilibrium point. One can see that the Nash equilibrium point (6,4,2) is locally stable for small values of α. As parameter α increases, the equilibrium point becomes unstable, infinitely many period doubling bifurcation of the phase quantity behavior becomes chaotic. This means that for a large values of speed of adjustment of boundedly rational player, the dynamical game (12) converge always to complex dynamics.

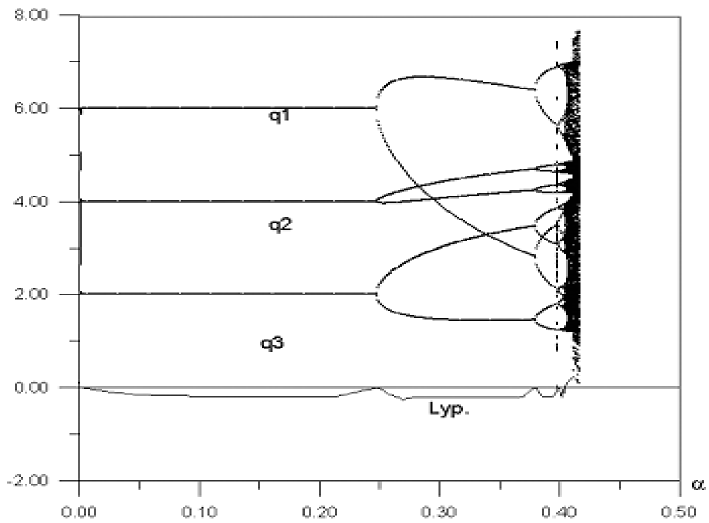

Figure 1: Bifurcation diagram of system (12) with respect to the parameter α. Also maximal Lyapunov exponents as a function of α is plotted.

Figure 2 represents the graph of a strange attractor of the dynamical game (12) for the parameter values $(a,b,c_1,c_2,c_3,\alpha,\beta)=$ (10, 0.5, 1, 2, 3, 0.41, 0.33) which exhibits fractal structure.

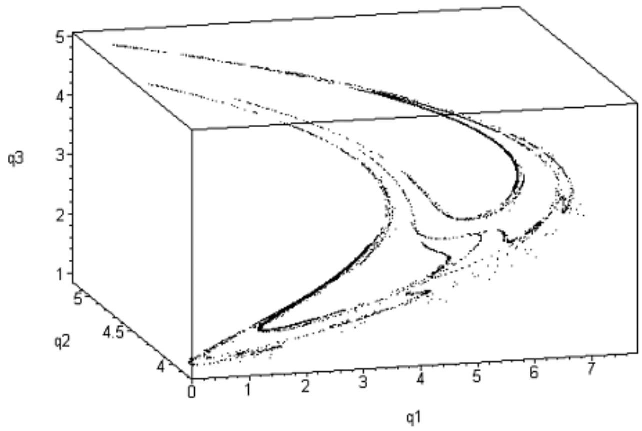

Figure 2: Strange attractor of system (12).

A bifurcation diagram with respect to the speed of adjustment of the second player,β, (adaptive player) and maximal Lyapunov exponent are shown in Figure 3.

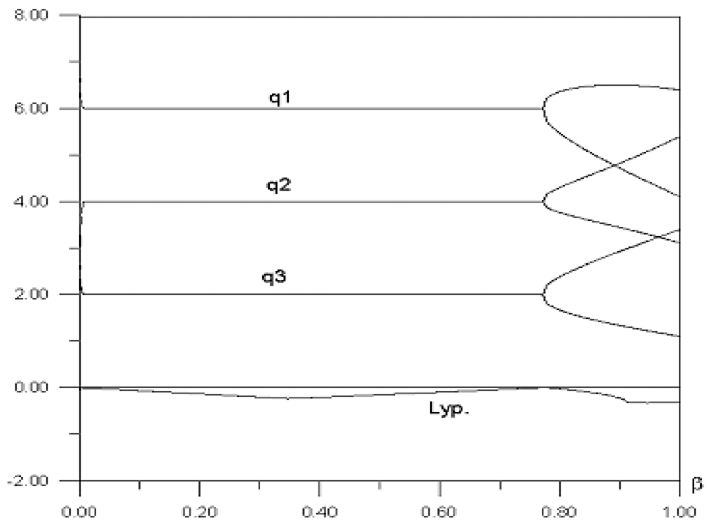

Figure 3: Bifurcation diagram of system (12) with respect to the parameter β. Also maximal Lyapunov exponents as a function of is plotted.

From the present analyses one can see that when all parameters kept fixed and only varied the speed of adjustment of boundedly rational player the structure of the market of triopoly game becomes complicated through period doubling bifurcations. In addition, more complex bounded attractors are created around Nash equilibrium point, which are a periodic cycles of high order or chaotic attractor. Also the dynamics of the triopoly game is so complicated, consequently, the players are unable to again a complete understanding structure of the market. Hence complex dynamics depend on the parameter α means that existence of boundedly rational player in the market leads to more rich dynamics than the homogeneous case. So the heterogeneity or diversity of players belief can lead to a rich dynamics and more complexity.

4.1 Sensitive dependence on initial conditions

To demonstrate the sensitivity to initial conditions of system (12), we compute two orbits with initial points $(q_1(0), q_2(0), q_3(0))$ and $(q_1(0) + 0.0001, q_2(0), q_3(0))$, respectively. The compositional results is shown in Fig. 4. From the this figure it is clear that, at the beginning, the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

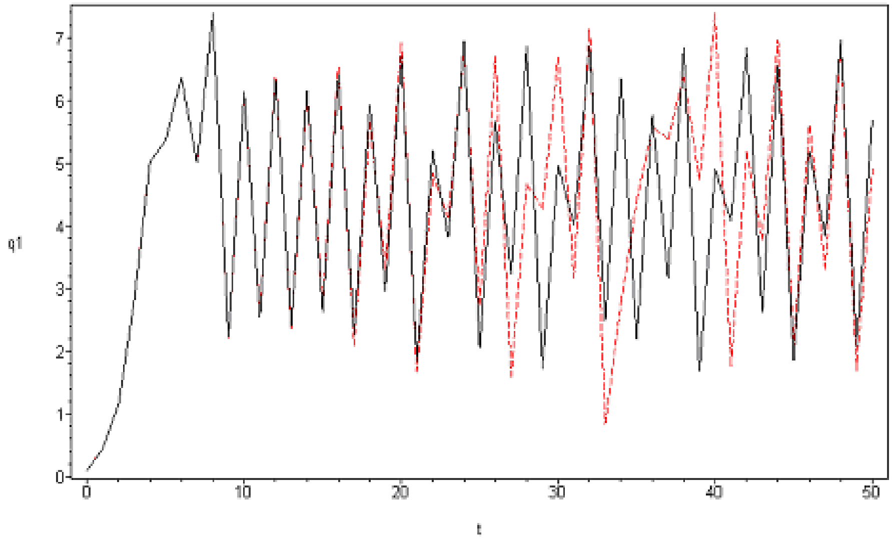

Figure 4: Sensitive dependence on initial conditions, q_1-coordinates of the two orbits, for system (12).

In addition, Fig. 4. shows sensitive dependence on initial conditions, q_1- coordinate of the two orbits, for system (12), is plotted against the time with the parameter constellation $(a, b, c_1, c_2, c_3, \alpha, \beta) = (10, 0.5, 1, 2, 3, 0.41, 0.33)$. The q_1-coordinates of initial conditions differ by 0.0001 and the other coordinates are kept equal.

4.2 Fractal dimension of the our map

It became well known that strange attractors are typically characterized by fractal dimensions. We examine the important characteristic of neighboring chaotic orbits to see how rapidly they separate each other. The Lyapunov dimension [14, 15] is defined as follows

$$\mathrm{d_L} = j + \frac{\sum_{i=1}^{i=j} \Lambda_i}{|\Lambda_j|},$$

with Λ_1, Λ_2, ..., Λ_n are Lyapunov Exponents where Λj is the largest integer such that $\sum_{i=1}^{i=j} \Lambda_i \geq 0$ and $\sum_{i=1}^{i=j+1} \Lambda_i < 0$. By the definition of the Lyapunov dimension and with help of the computer simulation one can show that the Lyapunov dimension is of the strange attractor of system (12). At the parameters values $(a, b, c_1, c_2, c_3, \alpha, \beta) = (10, 0.5, 1, 2, 3, 0.41, 0.33)$, three Lyapunov exponents exist and are $\Lambda_1 \approx 0.24$, $\Lambda_2 \approx 0.19$ and $\Lambda_3 \approx -2.7$. In our case of the three dimensional map has the Lyapunov dimension which is given by

$$d_{\mathrm{L}} = 2 + \frac{\Lambda_1 + \Lambda_2}{|\Lambda_3|}, \quad \Lambda_1 > \Lambda_2 > \Lambda_3$$

Therefore the map (12) has a fractal dimension $d_L \approx 2 + \frac{0.43}{2.7} = 2.1593$, which is hyperchaotic behavior and the same fractal dimension of Henon like map [16].

5 Conclusions

In this paper, we have presented triopoly game, which contains three-types of heterogeneous players: boundedly rational, adaptive and naive. We have explored the dynamics of the model under different regimes of the main parameters, such as the players speed of adjustment . Our analysis has shown that the heterogeneity or diversity of players behavior may lead to different dynamic scenarios, characterized by complex chaotic behavior.

References

[1] A. Cournot: Researches into the principles of the theory of wealth. *Engl. trans., chapter VII, Irwin Paper Back Classics in Economics*(1963).

[2] T. Puu: The Complex dynamics with three oligopolists. *Chaos Solitons Fractals*. 7 ,2075-2081(1996)

[3] H.N. Agiza, G.I. Bischi, M. Kopel: Multistability in a Dynamic Cournot Game with Three Oligopolists. *Math. Comput. Simulation.* 51,63-90(1999)

[4] H.N. Agiza: Explicit stability zones for Cournot games with 3 and 4 competitors. *Chaos Solitons Fractals*.9, 1955-1966(1998)

[5] A. Agliari, L. Gardini , T. Puu: The dynamics of a triopoly game. *Chaos Solitons Fractals*.11, 2531-2560(2000)

[6] W. J. Den-Haan: The importance of the number of different agents in a heterogeneous asset-pricing model. *Journal of Economic Dynamics and Control*. 25, 721-746(2001)

[7] A. Kirman, J. B. Zimmermann: Economics with heterogeneous Interacting Agents. *Lecture Notes in Economics and Mathematical Systems 503 (Springer, Berlin)* (2001)

[8] J.B. Rosser: The development of complex oligopoly dynamics theory. In Text *Book Oligopoly Dynamics : Models and Tools Springer -Verleg (2002)*

[9] T. Onazaki, G. Sieg ,nd M. Yokoo: Stability, chaos and multiple attractors: A single agent makes a difference. *Journal of Economic Dynamics and Control* .27, 1917-1938(2003)

[10] H. N. Agiza, A. A. Elsadany: Nonlinear dynamics in the Cournot duopoly game with heterogeneous players. *Physica A*. 320,512-524(2003)

[11] H. N. Agiza, A. A. Elsadany: Chaotic dynamics in nonlinear duopoly game with heterogeneous players. *Applid Math, and computation.* 149, 843-860(2004)

[12] G.I. Bischi, M. Galletgatti , A. Naimazada: Symmetry-breaking bifurcations and representative firm in dynamic duopoly games. *Annals of Operations Resarch.* 89, 253-272(1999)

[13] S. N. Elaydi: An Introduction to Difference Equations. *Springer-Verlag Publishers* (1996)

[14] J.L. Kaplan , Y. A. Yorke : A regime observed in a fluid flow model of Lorenz. *Comm. Math. Phys*. 67,93-108(1979)

[15] J. H. E. Cartwright: Nonlinear stiffiness, lyapunov exponents, and attractor dimension. *Physics Letters A*. 264, 298-304(1999)

[16] K. Stefanski: Modelling chaos and Hyperchaos with 3-D maps. *Chaos Solitons & Fractals.* 9, 83-93(1998)