Hausdorff Dimension of Level Set Related to Symbolic System

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Abstract. This paper proves that the Hausdorff dimension of some level set related to symbolic system is less than 1.

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1 Introduction

Let $\Sigma_2$ denote the symbolic system $\prod_{n=0}^{\infty} \{-1,1\}$, that is,
$$\Sigma_2 = \{ x = x_0x_1\cdots x_n \cdots | x_i = -1 \text{ or } 1 \}.$$ 

Suppose $\Sigma_2$ is equipped with a metric $d$ defined by $d(x, y) = 2^{-m}$ with $m = \min\{ i : x_i \neq y_i \}$ for $x \neq y$, where $x = x_0x_1\cdots x_{n-1}\cdots$ and $y = y_0y_1\cdots y_{n-1}\cdots$.

Let $\sigma$ be the left shift operator on $\Sigma_2$ defined by $\sigma(x_0x_1\cdots x_{n-1}\cdots) = x_1\cdots x_{n-1}\cdots$. The cylinder $[x_0x_1\cdots x_{n-1}]$ is the set $\{ y = y_0y_1\cdots y_{n-1}\cdots | y_i = x_i \text{ for } 0 \leq i \leq n-1\}$. There is a measure $\mu$ such that
$$\mu([x_0x_1\cdots x_{n-1}]) = 2^{-n}$$ 
for any cylinder $[x_0x_1\cdots x_{n-1}]$. We use the notation $\dim_H$ to denote the Hausdorff dimension. In fact, we have
$$\dim_H \Sigma_2 = 1 \text{ (see [6]).}$$

Given a real sequence $\{b_n\}_{n=0}^{\infty}$ with $\lim_{n \to \infty} b_n = 0$ and $\sum_{n=0}^{\infty} |b_n| = +\infty$, for any $a \in \mathbb{R}$, let
$$E_a = \{ x \in \Sigma_2 : \sum_{n=0}^{\infty} b_nx_n = a \}.$$ 

A work of Beyer [1] got an lower estimation $\dim_H E_a \geq 1/2$ for each $a \in \mathbb{R}$ under the variational condition
$$\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty.$$ 

J. WU [5] proved that $\dim_H E_a = \dim_H \Sigma_2 = 1$ also under the variational condition. Furthermore, L. F. XI [7] proved that $\dim_H E_a = 1$ without the variational condition $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$. Please see [2-4] and [8-11] for related topics.

For any $a \in \mathbb{R}$, we have $E_a \subset \Delta$, where
$$\Delta = \{ x = x_0x_1\cdots \in \Sigma_2 : \sum_{n=0}^{\infty} b_nx_n \text{ converges} \}.$$

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In this paper, we will study the Hausdorff dimension of $\Delta$.

In [7], we have the following result: If $\{b_n\}_{n \geq 0}$ is a real sequence such that $\lim_{n \to \infty} b_n = 0$, then $\dim_H \Delta = 1$, i.e.,

$$\dim_H \{x \in \Sigma_2 : \sum_{n=0}^{\infty} b_n x_n \text{ converges} \} = 1.$$ 

Note that in this result of [7], we do not need the conditions $\sum_{n=0}^{\infty} |b_n| = +\infty$ and $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$.

Let $f : \Sigma_2 \to \mathbb{R}$ be defined by $f(x_0 x_1 \cdots x_{n-1} \cdots) = x_0$ for $x_0 x_1 \cdots x_{n-1} \cdots \in \Sigma_2$. Then we have

$$\sum_{n=0}^{\infty} b_n x_n = \sum_{n=0}^{\infty} b_n f(\sigma^n x),$$

and naturally the following claim may be true:

**Claim 1.** For any real sequence $\{b_n\}_{n \geq 0}$ with $b_n \to 0$ and any continuous function $f : \Sigma_2 \to \mathbb{R}$,

$$\dim_H \{x \in \Sigma_2 : \sum_{n=0}^{\infty} b_n f(\sigma^n x) \text{ converges} \} = 1.$$

*However* the above claim is not true. In fact, we can prove the following theorem, which shows that the integral of $f$ shall be zero for the validity of the above claim.

**Theorem 1.** There is a sequence $\{a_n\}_{n \geq 0}$ of positive numbers with $\lim_{n \to \infty} a_n = 0$ and $0 < a_{n+1} \leq a_n$ for all $n$, such that for any continuous function $f : \Sigma_2 \to \mathbb{R}$ satisfying $\int_{\Sigma_2} f \, d\mu \neq 0$,

$$\dim_H \{x \in \Sigma_2 : \sum_{n=0}^{\infty} a_n f(\sigma^n x) \text{ converges} \} < 1.$$

### 2 Preliminary

Suppose $f : \Sigma_2 \to \mathbb{R}$ is a continuous function.

**Lemma 1.** Suppose $f(x)$ only depends on the first $q$ digits $x_1 \cdots x_q$ of $x \in \Sigma_2$ satisfying $\int_{\Sigma_2} f \, d\mu = 0$. If $\delta > 0$, then there is a constant $0 < \tau < 1$ such that for any $n$,

$$\mu \{x : \sum_{i=0}^{n-1} f(\sigma^i x) \geq n\delta \} \leq O(\tau^n).$$

**Proof.** Without loss of generality, we may assume that $n \geq \lceil 4q \max_{y \in \Sigma_2} |f(y)|/\delta \rceil$.

Suppose $n = qk + p$, where $p \in \mathbb{N} \cap [0, q-1]$. Then $2qk \geq n \geq \lceil 4q \max_{y \in \Sigma_2} |f(y)|/\delta \rceil$, which implies $k \geq 2 \max_{y \in \Sigma_2} |f(y)|/\delta$. We have

$$\{x : \sum_{i=0}^{n-1} f(\sigma^i x) \geq n\delta \} \subset \{x : \sum_{i=0}^{qk-1} f(\sigma^i x) \geq n\delta - p \max_{y \in \Sigma_2} |f(y)|\} \subset \{x : \sum_{i=0}^{qk-1} f(\sigma^i x) \geq (qk)(\delta/2)\}.$$

Therefore, it suffices to prove that $\mu \{x : \sum_{i=0}^{qk-1} f(\sigma^i x) \geq (qk)(\delta/2)\} \leq O(\tau^{qk})$ for some constant $\tau \in (0, 1)$.

Then for any $0 \leq i \leq q-1$,

$$\{f \circ \sigma^i, f \circ \sigma^{i+q}, \cdots, f \circ \sigma^{i+(k-1)q}\}$$

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can be considered as an i.i.d. with expectation 0.

Let
\[ S_k^{(i)} = f \circ \sigma^i + f \circ \sigma^{i+q} + \cdots + f \circ \sigma^{i+(k-1)q} = \sum_{j=0}^{k-1} f \circ \sigma^{i+jq}. \]

Then it follows from Cramer Theorem of large deviation theory that
\[ \mu\{x: \frac{S_k^{(i)}}{k} \geq \delta/2\} = \exp(-\inf_{|x| \geq \delta/2} \Lambda^*(x)k + o(k)) \leq O(a^k), \]
for some \( a \in (0, 1) \), where legendre transformation \( \Lambda^*(x) \) only depends on \( f \).

On the other hand, we have
\[ \{x: \sum_{i=0}^{qk-1} f(\sigma^i x) \geq (qk)\delta/2\} \subseteq \bigcup_{i=0}^{q-1} \{x: \frac{S_k^{(i)}}{k} \geq \delta/2\}, \]
therefore,
\[ \mu\{x: \sum_{i=0}^{qk-1} f(\sigma^i x) \geq (qk)(\delta/2)\} \leq qO(a^k) \leq O(\tau^{qk}), \]
for constant \( \tau = a^{1/q} \in (0, 1) \). \( \square \)

3 Proof of Theorem 1

Suppose an integer sequence \( b_k \uparrow \infty \) satisfies \( b_{k-1}/b_k \rightarrow 0 \). Let \( d_k \) be a sequence with \( d_k \uparrow \infty \) and \( d_k/b_k \rightarrow 0 \). We define a new sequence \( a_n \) as follows: If \( n \in [b_k, b_{k+1}) \), set \( a_n = (d_k)^{-1} \). Then \( a_n \downarrow 0 \).

Let
\[ \tilde{f} = f - \int_{\Sigma_2} f \, d\mu, \]
then we have
\[ \int_{\Sigma_2} \tilde{f} \, d\mu = 0. \]

Given \( \varepsilon \in (0, \int_{\Sigma_2} f \, d\mu / 4) \), we approximate \( \tilde{f}(x) \) by some function \( \tilde{f}(x) \) whose value only depends on the first \( q \) digits of \( x \), such that
\[ \sup |\tilde{f}(x) - \tilde{f}(x)| \leq \varepsilon \]
and \( \int_{\Sigma_2} \tilde{f} \, d\mu = 0. \)

Fix \( y \in \Sigma_2 \). Suppose \( \sum_{n=0}^{\infty} a_n f(\sigma^n y) \) converges, then there is \( k(y) \in \mathbb{N} \) such that
\[ |\sum_{b_k \leq n < b_{k+1}} a_n f(\sigma^n y)| \leq \varepsilon, \]
whenever \( k \geq k(y) \), which implies
\[ \begin{align*}
| \sum_{b_k \leq n < b_{k+1}} a_n \tilde{f}(\sigma^n y)| &\geq | \sum_{b_k \leq n < b_{k+1}} a_n f(\sigma^n y)| - | \sum_{b_k \leq n < b_{k+1}} a_n f(\sigma^n y)| \\
&\geq [(d_k)^{-1} (b_{k+1} - b_k)] \int_{\Sigma_2} f \, d\mu | - \varepsilon,
\end{align*} \]
consequently,
\[ \begin{align*}
| \sum_{b_k \leq n < b_{k+1}} a_n \tilde{f}(\sigma^n y)| &\geq | \sum_{b_k \leq n < b_{k+1}} a_n \tilde{f}(\sigma^n y)| - \sum_{b_k \leq n < b_{k+1}} a_n [\tilde{f}(\sigma^n y) - \tilde{f}(\sigma^n y)] \\
&\geq [(d_k)^{-1} (b_{k+1} - b_k)] \left( \int_{\Sigma_2} f \, d\mu | - \varepsilon \right) - \varepsilon,
\end{align*} \]

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and thus, we have

\[ | \sum_{b_k \leq n < b_{k+1}} \tilde{f}(\sigma^n y) | \geq (b_{k+1} - b_k) \left( | \int_{\Sigma_2} f d\mu | - \varepsilon (1 + \frac{d_k}{b_{k+1} - b_k}) \right), \]

where \( \frac{d_k}{b_{k+1} - b_k} < 1 \) for any \( k \geq k_1 \) where integer \( k_1 \) is a constant. Let \( m_k = b_{k+1} - b_k \) with \( m_k \to \infty \) as \( k \to \infty \), \( \delta = | \int_{\Sigma_2} f d\mu | / 2 \), and \( z = \sigma^{b_k} y \). Then for any \( k \geq \max(k(y), k_1) \), we have

\[ | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta, \]

which implies

\[ \{ y : \sum_{n=0}^{\infty} a_n f(\sigma^n y) \text{ converges} \} \subseteq \bigcup_{n} \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \}. \]

It suffices to show that there is a constant \( s \in (0, 1) \) such that for any \( k \),

\[ \dim_{H} \left( \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \right) \leq s. \]

Let \( A_k \) be the set of all the words \( x_1 \cdots x_{m_k+q-1} \in \{-1, 1\}^{m_k+q-1} \) such that \( | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \) whenever \( z \) lies in the cylinder \( [x_1 \cdots x_{m_k+q-1}] \).

Then by Lemma 1, we have

\[ \sharp A_k = 2^{m_k+q-1} \cdot \mu \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \leq O((2\tau)^{m_k}). \]

Consider the set \( \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \), for any \( k \geq n \), we give a natural covering

\[ \bigcup_{(x_1 \cdots x_k) \in \{-1, 1\}^k, (y_1 \cdots y_{m_k+q-1}) \in A_k} [x_1 \cdots x_k y_1 \cdots y_{m_k+q-1}]. \]

Thus for any \( s \in (\log(2\tau) / \log 2, 1) \),

\[ (2^{b_k} \sharp A_k) [2^{-\sigma^{-b_k+1+q-1}}] \leq O((2\tau)^{m_k} 2^{b_k - \sigma(b_k+1+q-1)}) \leq O(1), \]

since \( b_k/b_{k+1} \to 0 \) and \( m_k/b_{k+1} \to 1 \).

Therefore,

\[ H^s \left( \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \right) \leq O(1), \]

which implies

\[ \dim_{H} \left[ \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \right] \leq \log 2\tau / \log 2. \]

Consequently, we have

\[ \dim_{H} \{ y \in \Sigma_2 : \sum_{n=0}^{\infty} a_n f(\sigma^n y) \text{ converges} \} \]

\[ \leq \dim_{H} \bigcup_{n} \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \]

\[ \leq \sup_{n} \dim_{H} \left[ \bigcap_{k \geq n} \sigma^{-b_k} \{ z : | \sum_{i=0}^{m_k-1} \tilde{f}(\sigma^i z) | \geq m_k \delta \} \right] \]

\[ \leq \log 2\tau / \log 2 < 1. \]
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References


