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Hausdorff Dimension of Level Set Related to Symbolic System

Qiuli Guo [1]
Institute of Mathematics, Zhejiang Wanli University, Ningbo, 315100, P. R. China
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Abstract. This paper proves that the Hausdorff dimension of some level set related to symbolic system is less than 1.

Keywords: fractal; Hausdorff dimension; level set; symbolic system

1 Introduction

Let Σ_2 denote the symbolic system $\prod_{n=0}^{\infty}\{-1,1\}$, that is,

$$\Sigma_2 = \{x = x_0x_1\cdots x_n\cdots \mid x_i = -1 \text{ or } 1\}.$$

Suppose Σ_2 is equipped with a metric d defined by $d(x,y) = 2^{-m}$ with $m = \min\{i : x_i \neq y_i\}$ for $x \neq y$, where $x = x_0x_1\cdots x_{n-1}\cdots$ and $y = y_0y_1\cdots y_{n-1}\cdots$.

Let σ be the left shift operator on Σ_2 defined by $\sigma(x_0x_1\cdots x_{n-1}\cdots) = x_1\cdots x_{n-1}\cdots$. The cylinder $[x_0x_1\cdots x_{n-1}]$ is the set $\{y = y_0y_1\cdots y_{n-1}\cdots \mid y_i = x_i \text{ for } 0 \leq i \leq n-1\}$. There is a measure μ such that

$$\mu([x_0x_1\cdots x_{n-1}]) = 2^{-n}$$

for any cylinder $[x_0x_1\cdots x_{n-1}]$. We use the notation $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension. In fact, we have

$$\dim_{\mathrm{H}} \Sigma_2 = 1 \text{ (see [6]).}$$

Given a real sequence $\{b_n\}_{n\geq 0}$ with $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty}|b_n| = +\infty$, for any $a \in \mathbb{R}$, let

$$E_a = \{x \in \Sigma_2 : \sum_{n=0}^{\infty} b_n x_n = a\}.$$

A work of Beyer [1] got an lower estimation $\dim_{\mathrm{H}} E_a \geq 1/2$ for each $a \in \mathbb{R}$ under the variational condition

$$\sum_{n=0}^{\infty}|b_{n+1} - b_n| < \infty.$$

J. WU [5] proved that $\dim_{\mathrm{H}} E_a = \dim_{\mathrm{H}} \Sigma_2 = 1$ also under the variational condition. Furthermore, L. F. XI [7] proved that $\dim_{\mathrm{H}} E_a = 1$ without the variational condition $\sum_{n=0}^{\infty}|b_{n+1} - b_n| < \infty$. Please see [2-4] and [8-11] for related topics.

For any $a \in \mathbb{R}$, we have $E_a \subset \Delta$, where

$$\Delta = \{x = x_0x_1\cdots \in \Sigma_2 : \sum_{n=0}^{\infty} b_n x_n \text{ converges}\}.$$

[1] **Corresponding author**.Tel.: +86-574-8822 2230; Fax: +86-574-8822 2090. *E-mail address*: guoqiuli@zwu.edu.cn

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In this paper, we will study the Hausdorff dimension of Δ.

In [7], we have the following result: If $\{b_n\}_{n\geq 0}$ is a real sequence such that $\lim_{n\to\infty} b_n = 0$, then $\dim_H \Delta = 1$, i.e.,

$$\dim_{\mathrm{H}}\{x \in \Sigma_2 : \sum_{n=0}^{\infty} b_n x_n \text{ converges}\} = 1.$$

Note that in this result of [7], we do not need the conditions $\sum_{n=0}^{\infty} |b_n| = +\infty$ and $\sum_{n=0}^{\infty} |b_{n+1} - b_n| < \infty$.

Let $f : \Sigma_2 \to \mathbb{R}$ be defined by $f(x_0 x_1 \cdots x_{n-1} \cdots) = x_0$ for $x_0 x_1 \cdots x_{n-1} \cdots \in \Sigma_2$. Then we have

$$\sum_{n=0}^{\infty} b_n x_n = \sum_{n=0}^{\infty} b_n f(\sigma^n x),$$

and naturally the following claim *may be* true:

Claim 1. For any real sequence $\{b_n\}_{n\geq 0}$ with $b_n \to 0$ and any continuous function $f : \Sigma_2 \to \mathbb{R}$,

$$\dim_H\{x \in \Sigma_2 : \sum_{n=0}^{\infty} b_n f(\sigma^n x) \textit{ converges}\} = 1.$$

However the above claim is *not true*. In fact, we can prove the following theorem, which shows that the integral of f shall be zero for the validity of the above claim.

Theorem 1. *There is a sequence $\{a_n\}_{n\geq 0}$ of positive numbers with $\lim_{n\to\infty} a_n = 0$ and $0 < a_{n+1} \leq a_n$ for all n, such that for any continuous function $f : \Sigma_2 \to \mathbb{R}$ satisfying $\int_{\Sigma_2} f d\mu \neq 0$,*

$$\dim_H\{x \in \Sigma_2 : \sum_{n=0}^{\infty} a_n f(\sigma^n x) \textit{ converges}\} < 1.$$

2 Preliminary

Suppose $f : \Sigma_2 \to \mathbb{R}$ is a continuous function.

Lemma 1. *Suppose $f(x)$ only depends on the first q digits $x_1 \cdots x_q$ of $x \in \Sigma_2$ satisfying $\int_{\Sigma_2} f d\mu = 0$. If $\delta > 0$, then there is a constant $0 < \tau < 1$ such that for any n,*

$$\mu\{x : |\sum_{i=0}^{n-1} f(\sigma^i x)| \geq n\delta\} \leq O(\tau^n).$$

Proof. Without loss of generality, we may assume that $n \geq [4q \max_{y\in\Sigma_2} |f(y)|]/\delta$.

Suppose $n = qk + p$, where $p \in \mathbb{N} \cap [0, q-1]$. Then $2qk \geq n \geq [4q \max_{y\in\Sigma_2} |f(y)|]/\delta$, which implies $k \geq 2 \max_{y\in\Sigma_2} |f(y)|/\delta$. We have

$$\begin{aligned}\{x : |\sum_{i=0}^{n-1} f(\sigma^i x)| \geq n\delta\} &\subset \{x : |\sum_{i=0}^{qk-1} f(\sigma^i x)| \geq n\delta - p \max_{y\in\Sigma_2} |f(y)|\} \\ &\subset \{x : |\sum_{i=0}^{qk-1} f(\sigma^i x)| \geq (qk)(\delta/2)\}.\end{aligned}$$

Therefore, it suffices to prove that $\mu\{x : |\sum_{i=0}^{qk-1} f(\sigma^i x)| \geq (qk)(\delta/2)\} \leq O(\tau^{qk})$ for some constant $\tau \in (0, 1)$.

Then for any $0 \leq i \leq q-1$,

$$\{f \circ \sigma^i, f \circ \sigma^{i+q}, \cdots f \circ \sigma^{i+(k-1)q}\}$$

can be considered as an i.i.d. with expectation 0.

Let

$$S_k^{(i)} = f\circ\sigma^i + f\circ\sigma^{i+q} + \cdots + f\circ\sigma^{i+(k-1)q} = \sum_{j=0}^{k-1} f\circ\sigma^{i+jq}.$$

Then it follows from *Cramer Theorem* of large deviation theory that

$$\mu\{x : |\frac{S_k^{(i)}}{k}| \geq \delta/2\} = \exp(-\inf_{|x|\geq\delta/2}\Lambda^*(x)k + o(k)) \leq O(a^k),$$

for some $a\in(0,1)$, where legendre transformation $\Lambda^*(x)$ only depends on f.

On the other hand, we have

$$\{x : |\sum_{i=0}^{qk-1} f(\sigma^i x)| \geq (qk)\delta/2\} \subset \bigcup_{i=0}^{q-1}\{x : |\frac{S_k^{(i)}}{k}| \geq \delta/2\},$$

therefore,

$$\mu\{x : |\sum_{i=0}^{qk-1} f(\sigma^i x)| \geq (qk)(\delta/2)\} \leq qO(a^k) \leq O(\tau^{qk}),$$

for constant $\tau = a^{1/q}\in(0,1)$. $\square$

3 Proof of Theorem 1

Suppose an integer sequence $b_k\uparrow\infty$ satisfies $b_{k-1}/b_k\to 0$. Let d_k be a sequence with $d_k\uparrow\infty$ and $d_k/b_k\to 0$. We define a new sequence a_n as follows: If $n\in[b_k, b_{k+1})$, set $a_n = (d_k)^{-1}$. Then $a_n\downarrow 0$.

Let

$$\bar f = f - \int_{\Sigma_2} f d\mu,$$

then we have

$$\int_{\Sigma_2}\bar f d\mu = 0.$$

Given $\varepsilon\in(0, |\int_{\Sigma_2} f d\mu|/4)$, we approximate $\bar f(x)$ by some function $\tilde f(x)$ whose value only depends on the first q digits of x, such that

$$\sup|\bar f(x) - \tilde f(x)| \leq \varepsilon \text{ and } \int_{\Sigma_2}\tilde f d\mu = 0.$$

Fix $y\in\Sigma_2$. Suppose $\sum_{n=0}^{\infty} a_n f(\sigma^n y)$ converges, then there is $k(y)\in\mathbb{N}$ such that

$$|\sum_{b_k\leq n<b_{k+1}} a_n f(\sigma^n y)| \leq \varepsilon,$$

whenever $k\geq k(y)$, which implies

$$\begin{aligned} & |\textstyle\sum_{b_k\leq n<b_{k+1}} a_n\bar f(\sigma^n y)| \\ \geq\ & |\textstyle\sum_{b_k\leq n<b_{k+1}} a_n\int_{\Sigma_2} f d\mu| - |\textstyle\sum_{b_k\leq n<b_{k+1}} a_n f(\sigma^n y)| \\ \geq\ & [(d_k)^{-1}(b_{k+1}-b_k)|\textstyle\int_{\Sigma_2} f d\mu|] - \varepsilon, \end{aligned}$$

consequently,

$$\begin{aligned} |\sum_{b_k\leq n<b_{k+1}} a_n\tilde f(\sigma^n y)| &\geq |\sum_{b_k\leq n<b_{k+1}} a_n\bar f(\sigma^n y)| - \sum_{b_k\leq n<b_{k+1}} a_n|\bar f(\sigma^n y) - \tilde f(\sigma^n y)| \\ &\geq [(d_k)^{-1}(b_{k+1}-b_k)(|\int_{\Sigma_2} f d\mu| - \varepsilon)] - \varepsilon, \end{aligned}$$

and thus, we have

$$|\sum_{b_k\le n<b_{k+1}}\tilde{f}(\sigma^n y)|\ge (b_{k+1}-b_k)[|\int_{\Sigma_2} f d\mu|-\varepsilon(1+\frac{d_k}{b_{k+1}-b_k})],$$

where $|\frac{d_k}{b_{k+1}-b_k}|<1$ for any $k\ge k_1$ where integer k_1 is a constant. Let $m_k=b_{k+1}-b_k$ with $m_k\to\infty$ as $k\to\infty$, $\delta=|\int_{\Sigma_2} f d\mu|/2$, and $z=\sigma^{b_k}y$. Then for any $k\ge\max(k(y),k_1)$, we have

$$|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta,$$

which implies

$$\{y:\sum_{n=0}^{\infty}a_n f(\sigma^n y)\text{ converges}\}\subset\bigcup_n\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}.$$

It suffices to show that there is a constant $s\in(0,1)$ such that for any k,

$$\dim_{\mathrm{H}}[\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}]\le s.$$

Let A_k be the set of all the words $x_1\cdots x_{m_k+q-1}\in\{-1,1\}^{m_k+q-1}$ such that $|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta$ whenever z lies in the cylinder $[x_1\cdots x_{m_k+q-1}]$.

Then by Lemma 1, we have

$$\sharp A_k=2^{m_k+q-1}\cdot\mu\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}\le O((2\tau)^{m_k})$$

Consider the set $\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}$, for any $k\ge n$, we give a natural covering

$$\bigcup_{(x_1\cdots x_{b_k})\in\{-1,1\}^{b_k},(y_1\cdots y_{m_k+q-1})\in A_k}[x_1\cdots x_{b_k}y_1\cdots y_{m_k+q-1}].$$

Thus for any $s\in(\log(2\tau)/\log 2,1)$,

$$(2^{b_k}\sharp A_k)[2^{-(b_{k+1}+q-1)}]^s\le O((2\tau)^{m_k}2^{[b_k-s(b_{k+1}+q-1)]})\le O(1),$$

since $b_k/b_{k+1}\to 0$ and $m_k/b_{k+1}\to 1$.

Therefore,

$$H^s(\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\})\le O(1),$$

which implies

$$\dim_{\mathrm{H}}[\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}]\le\log 2\tau/\log 2.$$

Consequently, we have

$$\begin{aligned}
&\dim_{\mathrm{H}}\{y\in\Sigma_2|\sum_{n=0}^{\infty}a_n f(\sigma^n y)\text{ converges}\}\\
\le\ &\dim_{\mathrm{H}}[\bigcup_n\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}]\\
\le\ &\sup_n\dim_{\mathrm{H}}[\bigcap_{k\ge n}\sigma^{-b_k}\{z:|\sum_{i=0}^{m_k-1}\tilde{f}(\sigma^i z)|\ge m_k\delta\}]\\
\le\ &\log 2\tau/\log 2<1.
\end{aligned}$$

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