Solitary Wave Solutions and Double Periodic Solutions of the General Discrete mKdV Equation

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Abstract: A Sine-Gordon expansion method to construct new exact solutions of nonlinear differential-difference equations is presented. With the aid of symbolic computation, this method is used to seek more types of solitary wave solutions and double periodic solutions of the general discrete mKdV equation.

Keywords: Sine-Gordon expansion method; solitary wave solutions; double periodic solutions; discrete mKdV equation

1 Introduction

Nonlinear differential-difference equations (NDDEs) have played a significant role in nonlinear science and have received a lot of attention in the last 30 years or so. NDDEs, which are semi-discretized with some (or all) of their spatial variables discretized while time is usually kept continuous, are important in modeling such physical phenomena as particle vibrations in lattices, charge fluctuations in network, pulses in biological chains, etc. One can say that NDDEs occur whenever discrete phenomena are studied.

There are many studies about solitary wave solutions to partial differential equations (PDEs). While researches on NDDEs are relatively limited. In 2004, Baldwin et al. [6] extended the continuous tanh function method to NDDEs and obtained solitary wave solutions (hyperbolic tangent functions) of some discrete soliton equations. Since then, many modified method appeared. For example, the sine-cosine expansion method, modified hyperbolic function method, etc [5, 4, 3] Those works allow one to directly derive many different types of travelling wave solutions of DDEs.

We will present some new explicit solitary wave solutions of the general discrete mKdV equation by using the discrete Sine-Gordon expansion method. The model we consider is

\[
\frac{d u_n(t)}{dt} = (\alpha + \beta u_n + \gamma u_n^2)(u_{n+1} - u_{n-1}),
\]

where \(\alpha, \beta\) and \(\gamma \neq 0\) are constants. Eq.(1) can be seen as a discrete version of the nonlinear partial differential equation:

\[
u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{x,x,x} = 0,
\]

which has been widely studied. Eq. (1) contains the following lattice equations, namely:

(1) Discrete mKdV lattice

\[
\frac{d u_n(t)}{dt} = (\alpha - u_n^2)(u_{n+1} - u_{n-1}).
\]
(2) Volterra lattice equation
\[ \frac{du_n(t)}{dt} = u_n(u_{n-1} - u_{n+1}). \] (3)

(3) Modified Volterra lattice
\[ \frac{du_n(t)}{dt} = u_n^2(u_{n+1} - u_{n-1}). \] (4)

(4) Hybrid lattice
\[ \frac{du_n(t)}{dt} = (1 + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}). \] (5)

The polynomial travelling wave solutions in tanh of (2) and (5) are given in [6]. The explicit solutions in sinh-cosh form can be found in [4]. (3) and (4) are studied in [3].

2 Discrete Sine-Gordon expansion method

We describe the discrete Sine-Gordon expansion method [3] in detail in this section. It contains four steps.

Consider the following 1+1 dimensional nonlinear polynomial DDE,
\[ \Delta(u_{n+p1}(t), u_{n+p2}(t), \cdots, u_{n+p_k}(t), u'_{n+p1}(t), u'_{n+p2}(t), \cdots, u'_{n+p_k}(t), \cdots, u^{(r)}_{n+p1}(t), u^{(r)}_{n+p2}(t), \cdots, u^{(r)}_{n+p_k}(t) = 0, \] (6)

where the dependent variable \( u_n \) is a function of the variable \( n \) and the time variable \( t \). \( t \) is continuous while \( n, p_i \) is in \( Z \). And \( u^{(r)}(t) \) denotes derivative terms of order \( r \).

We seek solutions in the traveling frame of reference,
\[ u_n(t) = U(\xi_n), \xi_n = kn + ct + \xi_0, \] (7)

where the coefficients \( k, c \) and the phase \( \xi_0 \) are all constants.

Then
\[ u_{n+p_j} = U(\xi_n + \delta_j) \] (8)

where \( \delta_j = kp_j \)

**Step 1:** We assume that (6) has the solution in the form
\[ u_n(t) = U(\omega(\xi_n)) = A_0 + \sum_{i=1}^{i=N} \frac{\sin^{i-1}[\omega(\xi_n)][A_i\sin\omega(\xi_n) + B_i\cos\omega(\xi_n)]}{[P + Q\sin\omega(\xi_n) + R\cos\omega(\xi_n)]^i} \] (9)

where \( A_i, B_i, N, P, Q, R, P^2 + Q^2 + R^2 \neq 0 \) are constants to be determined later and \( \omega(\xi_n) \) satisfies
\[ \frac{d\omega(\xi_n)}{d\xi_n} = \pm[a + b\sin^2\omega(\xi_n)]^{\frac{1}{2}} \] (10)

We first determine the degree \( N \) by balancing the highest differential term and the nonlinear term of \( u_n \).

In order to deduce the expressions of these terms \( U(\xi_n + \delta_j) \) we need to consider them in the following cases.

**Case 1:** When \( a = 1, b = -1 \), Eq.(10) has solutions
\[ \sin[\omega(\xi_n)] = \tanh(\xi_n) \quad \text{or} \quad \cos[\omega(\xi_n)] = \sech(\xi_n) \] (11a)
\[ \sin[\omega(\xi_n)] = \coth(\xi_n) \quad \text{or} \quad \cos[\omega(\xi_n)] = \csch(\xi_n), \ t^2 = -1. \] (11b)

Note that
\[ \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \]
It can be seen that
\[ u_{n+p_j} = U(\xi_n + \delta_j) = U(\xi_n) \]

where
\[ \sin(\omega) = \frac{\sin(\omega(\xi_n) + \tanh(\delta_j))}{1 + \sin(\omega(\xi_n) \tanh(\delta_j))}, \quad \cos(\omega) = \frac{\cos(\omega(\xi_n) \sech(\delta_j))}{1 + \sin(\omega(\xi_n) \tanh(\delta_j))} \]

\[ = A_0 + \sum_{i=1}^{n} \left[ \frac{\sin(\omega(\xi_n) + \tanh(\delta_j))}{1 + \sin(\omega(\xi_n) \tanh(\delta_j))} \right] ^{i-1} \left[ A_1 \frac{\sin(\omega(\xi_n) + \tanh(\delta_j))}{1 + \sin(\omega(\xi_n) \tanh(\delta_j))} + B_1 \frac{\cos(\omega(\xi_n) \sech(\delta_j))}{1 + \sin(\omega(\xi_n) \tanh(\delta_j))} \right] \]

\[ (12) \]

Case 2: When \( a = 1 \) and \( b = -m^2 \), Eq.(10) has solutions
\[ \sin[\omega(\xi_n)] = sn(\xi_n, m) \quad \text{or} \quad \cos[\omega(\xi_n)] = cn(\xi_n, m) \]

where \( sn, cn \) are Jacobi elliptic functions and \( m \) is the modulus. Note that
\[ sn(x + y, m) = \frac{sn(x, m)cn(y, m)dn(y, m) + sn(y, m)cn(x, m)dn(x, m)}{1 - m^2sn(x, m)^2sn(y, m)^2} \]
\[ cn(x + y, m) = \frac{cn(y, m)cn(x, m) + sn(y, m)dn(y, m)sn(x, m)dn(x, m)}{1 - m^2sn(x, m)^2sn(y, m)^2} \]
\[ dn(x, m) = \sqrt{1 - m^2sn^2(x, m)} \]

We can see that \( U(\xi_n + \delta_j) \) are of the forms

\[ U(\xi_n + \delta_j) = U(\xi_n) \] \[ \sin(\xi_n) = E_1, \cos(\xi_n) = E_2 \]

where
\[ E_1 = \frac{\sin(\omega(\xi_n))cn(\delta_j, m)dn(\delta_j, m) + \cos(\omega(\xi_n))\sqrt{1 - m^2\sin^2(\omega(\xi_n))sn(\delta_j, m)}}{1 - m^2\sin^2(\omega(\xi_n))sn^2(\delta_j, m)}, \]
\[ E_2 = \frac{\cos(\omega(\xi_n))cn(\delta, m) - \sin(\omega(\xi_n))\sqrt{1 - m^2\sin^2(\omega(\xi_n))sn(\delta, m)dn(\delta, m)}}{1 - m^2\sin^2(\omega(\xi_n))sn^2(\delta, m)}, \]

Case 3: When \( a = m^2 \) and \( b = -1 \), Eq.(10) has solutions
\[ \sin[\omega(\xi_n)] = msn(\xi_n, m) \quad \text{or} \quad \cos[\omega(\xi_n)] = dn(\xi_n, m) \]

We can see that \( U(\xi_n + \delta_j) \) are of the forms

\[ U(\xi_n + \delta_j) = U(\xi_n) \] \[ \sin(\xi_n) = E_3, \cos(\xi_n) = E_4 \]

where
\[ E_3 = \frac{\sin(\omega(\xi_n))cn(\delta_j, m)dn(\delta_j, m) + \cos(\omega(\xi_n))\sqrt{1 - m^2\sin^2(\omega(\xi_n))sn(\delta_j, m)}}{1 - m^2\sin^2(\omega(\xi_n))sn^2(\delta_j, m)}, \]
\[ E_4 = \frac{\cos(\omega(\xi_n))dn(\delta, m) - \sin(\omega(\xi_n))\sqrt{1 - m^2\sin^2(\omega(\xi_n))sn(\delta, m)cn(\delta, m)}}{1 - m^2\sin^2(\omega(\xi_n))sn^2(\delta, m)}, \]

Step 2: Substitute Eq.(9), Eq.(10), Eq.(11) and Eq.(12) into Eq.(6) and take numerator to get a polynomial equation in \( \omega^s \sin^i \omega \cos^j \omega \).

Remark: When Case 2 or Case 3 is considered, we should use the corresponding equations.

Step 3: Set the coefficients of \( \omega^s \sin^i \omega \cos^j \omega \) to zero to get a set of algebraic equations with respect to the unknowns \( k, c, P, Q, R, A_1, \) and \( B_1 \). Then solve the algebraic equations.

Step 4: Derive the solitary wave solutions, singular solitary solutions and doubly periodic solutions of the given NDDE.
3 New solutions of Eq. (1)

By balancing the highest order linear term with nonlinear terms, we get \( N = 1 \). Thus we assume that \( U(\xi_n) \) takes the form

\[
U(\xi_n) = A_0 + \frac{A_1 \sin \omega(\xi_n) + B_1 \cos \omega(\xi_n)}{P + Q \sin \omega(\xi_n) + R \cos \omega(\xi_n)}
\]  

(19)

where \( A_0, A_1, B_1, P, Q, R \) are constants to be determined later and \( \omega(\xi_n) \) satisfied Eq.(10). We use the form of \( U(\xi_n \pm k) \) according to Eq.(12) or Eq.(14) or Eq.(17) in three cases. By Step 1 2, and 3 given in the above section, we obtain the following explicit solutions of Eq. (1):

\[
u_{n,1} = -\frac{\beta}{2\gamma} \pm \frac{\beta^2 - 4\alpha \gamma}{2\gamma} \sinh(k) \text{csch} \left( kn - \frac{\sinh(k) (\beta^2 - 4\alpha \gamma)}{2\gamma} t + \xi_0 \right)
\]  

(20)

\[
u_{n,2} = -\frac{\beta}{2\gamma} \pm \frac{\beta^2 - 4\alpha \gamma}{2\gamma} \sinh(k) \text{sech} \left( kn - \frac{\sinh(k) (\beta^2 - 4\alpha \gamma)}{2\gamma} t + \xi_0 \right)
\]  

(21)

\[
u_{n,3} = -\frac{\beta}{2\gamma} \pm 4\alpha \gamma - \beta^2 \sinh(k) \text{sech} \left( kn + \frac{\sinh(k) (4\alpha \gamma - \beta^2)}{2\gamma} t + \xi_0 \right)
\]  

(22)

\[
u_{n,4} = -\frac{\beta}{2\gamma} \pm 4\alpha \gamma - \beta^2 \sinh(k) \text{csch} \left( kn + \frac{\sinh(k) (4\alpha \gamma - \beta^2)}{2\gamma} t + \xi_0 \right)
\]  

(23)

\[
u_{n,5} = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{4\alpha \gamma - \beta^2}}{2\gamma} \frac{\sqrt{P^2 - Q^2} \sinh(k) \text{sech}(kn + \frac{\sinh(k)(4\alpha \gamma - \beta^2)}{2\gamma} t + \xi_0)}{P + Q \tanh(kn + \frac{\sinh(k)(4\alpha \gamma - \beta^2)}{2\gamma} t + \xi_0)}
\]  

(24)

\[
u_{n,6} = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{4\alpha \gamma - \beta^2}}{2\gamma} \frac{\sqrt{P^2 - Q^2} \sinh(k) \text{csch}(kn + \frac{\sinh(k)(4\alpha \gamma - \beta^2)}{2\gamma} t + \xi_0)}{P + Q \coth(kn + \frac{\sinh(k)(4\alpha \gamma - \beta^2)}{2\gamma} t + \xi_0)}
\]  

(25)

\[
u_{n,7} = \frac{2\alpha(e^k + 1)(\cosh k - 1) \text{sech}(kn + 2\alpha \sinh k t + \xi_0)}{2\sqrt{\gamma \alpha(e^{2k} + 1) + (\beta^2 - 2\gamma \alpha)e^k + (e^k + 1)\beta \text{sech}(kn + 2\alpha \sinh k t + \xi_0)}}
\]  

(26)

\[
u_{n,8} = \frac{2\alpha(e^k + 1)(\cosh k - 1) \text{csch}(kn + 2\alpha \sinh k t + \xi_0)}{2\sqrt{\gamma \alpha(e^{2k} + 1) + (\beta^2 - 2\gamma \alpha)e^k + (e^k + 1)\beta \text{csch}(kn + 2\alpha \sinh k t + \xi_0)}}
\]  

(27)

\[
u_{n,9} = \frac{2\alpha(e^k + 1)(\cosh k - 1) \text{sech}(\xi_n)}{2\sqrt{(2\gamma \alpha - \beta^2)e^k - \gamma \alpha(e^{2k} + 1)\tanh(\xi_n) + (e^k + 1)\beta \text{sech}(\xi_n)}}
\]  

(28)

where \( \xi_n = kn + 2\alpha \sinh kt + \xi_0 \)

\[
u_{n,10} = \frac{2\alpha(e^k + 1)(\cosh k - 1) \text{csch}(\xi_n)}{2\sqrt{(2\gamma \alpha - \beta^2)e^k - \gamma \alpha(e^{2k} + 1)\coth(\xi_n) + (e^k + 1)\beta \text{csch}(\xi_n)}}
\]  

(29)

where \( \xi_n = kn + 2\alpha \sinh kt + \xi_0 \)

\[
u_{n,11} = \frac{2\alpha(e^k + 1)(\cosh k - 1) \text{sech}(\xi_n)}{P + (e^k + 1)\beta(Q \tanh(\xi_n) + R \text{sech}(\xi_n))}
\]  

(30)

where \( P = \pm \sqrt{Q^2 \beta^2(e^k + 1)^2 + R^2(4\alpha \gamma(e^k - 1)^2 + 4\beta^2e^k)} \), \( \xi_n = kn + 2\alpha \sinh kt + \xi_0 \)

\[
u_{n,12} = \frac{2\alpha(e^k + 1)(\cosh k - 1) \text{csch}(\xi_n)}{P + (e^k + 1)\beta(Q \coth(\xi_n) + R \text{csch}(\xi_n))}
\]  

(31)

where \( P = \pm \sqrt{Q^2 \beta^2(e^k + 1)^2 + R^2(4\alpha \gamma(e^k - 1)^2 + 4\beta^2e^k)} \), \( \xi_n = kn + 2\alpha \sinh kt + \xi_0 \)

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\[ u_{n,13} = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} \text{msn}(k, m)\text{sn}(kn - \frac{\text{sn}(k, m)(\beta^2 - 4\alpha\gamma)}{2\gamma}t + \xi_0, m) \] (32)

\[ u_{n,14} = -\frac{\beta}{2\gamma} + \sqrt{\frac{\beta^2 - 4\alpha\gamma}{4m^2\text{sn}^2(k, m)\gamma^2 - 4\gamma^2} \text{sn}(k, m)\text{mcn}(\xi_n, m)} \] (33)

\[ u_{n,15} = -\frac{\beta}{2\gamma} \pm \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\gamma} \text{sc}(k, m)\text{dn}(kn + \frac{4\alpha\gamma - \beta^2}{2\gamma} \text{sc}(k, m)t + \xi_0, m) \] (34)

Among the above solutions, some are solutions in complex field, i.e., (21), (23), (25), (27), (29), (31). Solutions (32), (33) and (34) are double periodic solutions. Others are travelling solitary wave solutions. Graphic of (22) is given in Fig.1 for parameters \( \alpha = 1, \beta = 1, \gamma = 2, k = 0.2 \). With the same parameters, Fig. 2 shows the solution profiles of (22) for different times \( t = -10, -5, 0, 5, 10 \).

![Figure 1: Graphic of (22), with parameters \( \alpha = 1, \beta = 1, \gamma = 2, k = 0.2 \)](image1)

![Figure 2: Profile of (22) for different times \( t = -10, -5, 0, 5, 10 \).](image2)

Taken parameters \( \alpha = 1, \beta = 1, \gamma = 2, k = 0.2, p = 3, q = 1 \), graphic of (24) is given in Fig.3. Fig. 4 shows the corresponding plain profile when \( t = 0 \).

![Figure 3: Graphic of (24) as \( \alpha = 1, \beta = 1, \gamma = 2, k = 0.2, P = 3, Q = 1 \).](image3)

![Figure 4: Plain profile of (24) when \( t = 0 \) with the same parameters in Fig.3](image4)

As to the double periodic solutions, we give the graphics of (32) as an example. Parameters are \( \alpha = 1, \beta = 3, \gamma = 1, m = 0.5, k = 0.2 \). Fig. 6 gives the solution profiles of (32) for different discrete variables \( n = 0, 5, 10 \).

4 Conclusions

We have displayed a discrete Sine-Gordon expansion method and applied it to the general discrete mKdV equation such that many types of solitary wave solutions and doubly periodic solutions are derived. The method can be applied to other discrete soliton equations.
Figure 5: Graphic of (32) when $\alpha = 1, \beta = 3, \gamma = 1, m = 0.5, k = 0.2$.

Figure 6: Graphic of (32) for different value of discrete variable ($n = 0, 5, 10$).

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