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Liapunov Methods for Error Estimate of Waveform Relaxation

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Abstract: In this paper we consider the error estimate of waveform relaxation. By use of the Liapunov functionals, we obtain a new judgment criterion for the convergence of error sequences. At the same time, we show that the new criterion is efficient with an example.

Key words: waveform relaxation; error estimate; Liapunov functionals
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1 Introduction

Waveform relaxation (WR) method has often been used to solve some large and complex systems on parallel computers. Such a method splits the systems into subsystems and evolves subsystems independently over a short period of time before communicating results to neighboring subsystems. There are special utilities on numerical calculating for differential equations. Therefore the study of global convergence of such methods is become more important. Earlier studies concentrated on this topic are mainly recur to Lipschitz condition, or Taylor expansions, or comparable theorems of differential equations etc., these methods are convenient for translating nonlinear equations into linear equations (or in-equations). But some originally characteristics of the iteration are ignored in the process. In this paper we will give a judgment criterion for the convergence of error sequence using a kind of Liapunov functional. Our study is based on following idea: transform the error iterative sequence $\{e^k(t)\}$, $t \in [0,T]$ to a function on$[0,1)$. So, the original error iterative differential equations (or differential inequalities) on $[0,T]$ are shifted into another differential equations (or differential inequalities) on $[0,1)$. Hence we only need to discuss the asymptotic stability of the last differential equations (or differential inequalities) .

2 Main Result

Considering the ordinary differential equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)),\ t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $x(t) \in R^n$, $f : [0,T] \times R^n \to R^n$ are continuous functions.

Select the splitting function F which satisfies

$$F(t, p(t), p(t)) = f(t, p(t)), \tag{2.2}$$

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Then the waveform relaxation iterative form of the initial value problem (2.1) is

$$\begin{cases} \dot{x}^{k+1}(t) = F(t, x^{k+1}(t), x^k(t)), t \in [0, T], \\ x^{k+1}(0) = x_0, k = 0, 1, 2, \cdots \end{cases} \tag{2.3}$$

Let $e^k(t) = x^k(t) - x(t)$,where $x(t)$is the solution of equation (2.1), $x^k(t)$is the kth iteration of (2.3) respectively.

Under some conditions of F,we can conclude

$$\begin{cases} \dot{e}^{k+1}(t) \le g(t, e^{k+1}(t), e^k(t)), t \in [0, T], \\ e^{k+1}(0) = 0, k = 0, 1, 2, \cdots \end{cases} \tag{2.4}$$

or

$$\begin{cases} \frac{d}{dt}|e^{k+1}(t)| \le g(t, |e^{k+1}(t)|, |e^k(t)|), t \in [0, T], \\ e^{k+1}(0) = 0, k = 0, 1, 2, \cdots \end{cases} \tag{2.4$'$}$$

First we give the following hypothesis.

(H) The mapping g takes $[0, T] \times (bounded\ sets\ of\ C \times C)$ into bounded sets of R^n,$C = C([0, T], R^n)$.

Now we discuss the problem that the solution $\{e^{k+1}(t)\}_{k=1,2,\cdots}$ of (2.4) uniformly converges to zero(similarly for the problem with $(2.4)'$)

Let us define

$$y(t) = \begin{cases} e^0(t + T), & t \in [-T, 0), \\ e^1(t), & t \in [0, T), \\ e^2(t - T), & t \in [T, 2T), \\ \vdots & \vdots \end{cases} \tag{2.5}$$

Namely, $y(t) = e^{k+1}(t - kT), k = -1, 0, 1, 2, \cdots, t \in [kT, (k + 1)T)$

Let $I = [0, \infty) - \{kT\}_{k=0,1,2,\cdots}$

Then, $y(t)$is continuously differentiable in I,and satisfies the following differential inequalities.

$$\begin{cases} \dot{y}(t) \le \bar{g}(t, y(t), y(t - T)), t \in [0, \infty), \\ y(kT) = 0, k = -1, 0, 1, 2, \cdots, \end{cases} \tag{2.6}$$

where $\bar{g}(t, y(t), y(t - T)) = g(t - kT, y(t), y(t - T))$,$t \in [kT, (k + 1)T)$

Theorem 2.1. Let $u, v, \omega : R_+ \to R_+$ are continuous nondecreasing functions, $\omega(s)$is positive for $s > 0$.Under the hypothesis (H),if there is a Liapunov functional $V(t, \phi) : R_+ \times C_1 \to R_+$ such that

(1) $u(|\phi(0)|) \le V(t, \phi) \le v(|\phi|)$,

(2) $\dot{V}(t, y_t) \le -\omega(|y(t)|)$,

(3) $V(kT + 0, y_t) \le V(kT - 0, y_t)$,

then $\lim_{t\to+\infty} y(t) = 0$.Furthermore, $\lim_{k\to+\infty} e^{k+1}(t) = 0$ is holds uniformly for$t \in [0, T]$.

Proof: First we will prove that there exists a constant$L > 0$,such that $|\dot{y}(t)| \le L, t \in I.$
From the definitions of $y(t)$ and V, $V(t, y_t)$is right-continuous at $kT(k = 0, 1, 2, \cdots)$, and (2),(3) immediately implies that $V(t, y_t)$ is decreases monotonically on $[0, +\infty)$, Hence,

$$u(|y(t)|) \le V(t, y_t) \le V(T, y_T) \le v(|y_T|), t \in I, \tag{2.7}$$

that is, $|y(t)| \le u^{-1}(v(|y_T|)) \equiv M_1, t \in I,$

From the hypothesis(H),there is a constant$L > 0$, such that

$$|\dot{y}(t)| \le L, \quad t \in I \tag{2.8}$$

To prove $\lim_{t\to+\infty} y(t) = 0$, suppose that this is not the case. Then there exists a sequence $\{t_n\}_{n=1,2,\cdots}$, and $\sigma > 0$such that $|y(t_n)| \ge \sigma$.

From (2.8)we can easily find, $\exists\delta > 0$, such that

$$|y(t)| > \frac{\sigma}{2}, \ t \in [t_n - \delta, t_n + \delta] \subset I, (n = 1, 2, \cdots) \tag{2.9}$$

where $[t_n - \delta, t_n + \delta]$don't intersect for each other.

For every internals $[kT, (k+1)T)$ $(k = 1, 2, \cdots)$,condition(2) implies

$$V((k+1)T - 0, y_{(k+1)T}) - V(kT, y_{kT}) \leq -\int_{kT}^{(k+1)T} \omega(|y(t)|)dt$$

Combining condition(3),we obtain

$$V(t, y_t) - V(T, y_T) \leq -\int_T^t \omega(|y(t)|)dt, \ t \in I,$$

Therefore,

$$V(t, y_t) - V(T, y_T) \leq -\sum_{k=1}^{n} \int_{t_k-\delta}^{t_k+\delta} \omega(|y(t)|)dt,$$

here t is sufficiently large.

Notice that $\omega(s) > 0, s > 0$, and $\frac{\delta}{2} < |y(t)| < M_1, t \in [t_k - \delta, t_k + \delta]$.

There is a constant $M_2 > 0$,

$$\omega(|y(t)|) \geq M_2, t \in [t_k - \delta, t_k + \delta], k = 1, 2, \cdots$$

Hence,if t is sufficiently large,

$$\begin{aligned} V(t, y_t) - V(T, y_T) &\leq -\sum_{k=1}^{n} 2\delta M_2 \\ &= -2n\delta M_2, \end{aligned}$$

Select sufficiently large n, we have $V(t, y_t) < 0$,which is a contradiction. So, $\lim_{t \to +1} y(t) = 0$, and the proof is complete.

3 An example

Consider an iterative error sequence$\{e^k(t)\}$which satisfies

$$\begin{cases} \dot{e}^{k+1}(t) = -ae^{k+1}(t) + be^k(t)), t \in [0, T], \\ e^{k+1}(0) = 0, k = 0, 1, 2, \cdots \end{cases} \tag{3.1}$$

where $e^k(t) \in R, \ a, b$are constants,$a > 0$.

Theorem 3.1. If $|b| < a$, then $\lim_{k \to +1} e^k(t) = 0, t \in [0, T]$.

Proof: Let $V(y_t) = \frac{1}{2}y^2(t) + \frac{a}{2}\int_{t-T}^t y^2(t)dt$, where the definition of $y(t)$is the similar to that of Theorem 2.1.

Then V satisfies conditions (1),(3) of Theorem 2.1.

Now we will show that condition (2) of Theorem 2.1 is also hold.

$$\dot{V}(y_t) = y(t)\dot{y}(t) + \frac{a}{2}[y^2(t) - y^2(t - T)],$$

Notice that $y(t) = e^{k+1}(t - kT), y(t - T) = e^k(t), t \in [kT, (k+1)T)$.

Thus

$$\dot{e}^{k+1}(t - kT) = -ae^{k+1}(t - kT) + be^k(t - kT)),$$

Further,

$$\dot{y}(t) = -ay(t) + by(t - T) \tag{3.2}$$

Hence

$$\begin{aligned} \dot{V}(y_t) &= -ay^2(t) + by(t)y(t - T) + \frac{a}{2}[y^2(t) - y^2(t - T)] \\ &= -\frac{a}{2}y^2 + by(t)y(t - T) - \frac{a}{2}y^2(t - T). \quad (3.3) \end{aligned}$$

It is obvious that $-\frac{a}{2}y^2 + by(t)y(t-T) - \frac{a}{2}y^2(t-T)$ is negative definite,so the condition (2)of Theorem 2.1 is hold. Hence $\lim_{t \to +1} y(t) = 0$, that is $\lim_{k \to +1} e^k(t) = 0, t \in [0, T]$.

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