

# Image Measures and Statistical Mechanical Characterization for Their Dimensions

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**Abstract:** If the measures  $\mu$  and  $\nu$  are the images of  $\sigma$ -invariant ones in symbolic space, we can obtain the formulae of dimension of measures using entropy. Furthermore, we treat a class of Moran sets and give some statistical interpretation on dimensions and corresponding geometrical measures.

**Keywords:** dimension of measure; Hausdorff dimension; packing dimension; thermodynamic formalism; image of measure

## 1 Introduction

In recent years, the construction and existence of attractors have been widely mentioned in many fields of science solving a lot of problems especially on self-similar sets<sup>[1-3]</sup>. The chaotic attractors almost have fractal construction. In this paper, we mainly study a class of classical self-similar sets.

Associated with some geometrically defined measures, the Hausdorff dimension and the packing dimension are well known and often used. Though centered Hausdorff dimension does not exceed packing dimension generally, these two dimensions often coincide for most regular sets. In [4], the author investigated a class of Moran sets in general metric space, with respect to which the above two dimensions are different, while their corresponding measures are equivalent. In this paper, if the measures  $\mu$  and  $\nu$  in Theorems 4.2 and 4.3 are the image of  $\sigma$ -invariant ones in symbolic space, we can obtain similar formulae of dimension of measures using entropy as in the case of self-similar sets. In the last section, we treat a class of Moran sets and give some statistical interpretation to dimensions and corresponding geometrical measures.

## 2 The definition of measure and dimension of measure

We first recall the definition of the centered Hausdorff measure and the packing measure. Let  $S$  be a metric space,  $E \subseteq S$  and  $\delta > 0$ . A countable family  $\mathcal{B} = (B(x_i, r_i))_i$  of closed balls in  $S$  is called a centered  $\delta$ -covering of  $E$ . If  $E \subseteq \bigcup_i (B(x_i, r_i))$ ,  $x_i \in E$  and  $0 < r_i < \delta$  for all  $i$ . The family  $\mathcal{B}$  is called a centered  $\delta$ -packing of  $E$ . If  $x_i \in E$ ,  $0 < r_i < \delta$  and  $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$  for all  $i \neq j$ . We define the centered Hausdorff measure introduced by Raymond & Tricot.

Put  $\bar{C}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^s \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-covering of } E \right\}$ . The  $s$ -dimensional centered pre-Hausdorff measure  $\bar{C}^s(E)$  of  $E$  is defined by

$$\bar{C}^s(E) = \sup_{\delta > 0} \bar{C}_\delta^s(E).$$

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The set function  $\bar{C}^s$  is not necessarily monotone, and hence not necessarily an outer measure. But  $\bar{C}^s$  gives rise to a Borel measure, called the  $s$ -dimensional centered Hausdorff measure  $C^s(E)$  of  $E$ , and shown as follows:

$$C^s(E) = \sup_{F \subseteq E} \bar{C}^s(F).$$

We will now define the packing measure. Write

$$\bar{\mathcal{P}}_\delta^s(E) = \sup \left\{ \sum_{i=1}^{\infty} (2r_i)^s \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E \right\}.$$

The set function  $\bar{\mathcal{P}}^s$  is not necessarily countably subadditive, and hence not necessarily an outer measure. But  $\bar{\mathcal{P}}^s$  gives rise to a Borel measure, called the  $s$ -dimensional packing measure  $\mathcal{P}^s(E)$  of  $E$ , shown as follows:

$$\mathcal{P}^s(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \bar{\mathcal{P}}^s(E_i).$$

The packing measure was introduced by Taylor and Tricot using centered  $\delta$ -packing of open balls, and by Raymond and Tricot using centered  $\delta$ -packing of closed balls. From these two geometrical measures we can define the centered Hausdorff dimension  $\dim(E)$  and the packing dimension  $\text{Dim}(E)$ :

$$\dim(E) = \sup \{s \geq 0 : C^s(E) = \infty\}, \quad \text{Dim}(E) = \sup \{s \geq 0 : \mathcal{P}^s(E) = \infty\}.$$

In general  $\dim(E) \leq \text{Dim}(E)$  holds and it is easy to see that

$$\dim(E) = \inf \{s \geq 0 : C^s(E) = 0\}, \quad \text{Dim}(E) = \inf \{s \geq 0 : \mathcal{P}^s(E) = 0\}.$$

The following Theorem is useful and crucial.

**Theorem 2.1**<sup>[4]</sup> Let  $S$  be a metric space, and let  $\mu$  be a finite Borel measure. Let  $E \subseteq S$  be a Borel set. Then there exist four positive constants  $c_i, i = 1, 2, 3, 4$ , with which the following inequalities are true:

$$c_1 \mu(E) \inf_{x \in E} \left\{ \liminf_{r \rightarrow 0} \frac{r^s}{\mu(B(x, r))} \right\} \leq C^s(E) \leq c_2 \mu(E) \sup_{x \in E} \left\{ \liminf_{r \rightarrow 0} \frac{r^s}{\mu(B(x, r))} \right\}, \quad (2.1)$$

$$c_3 \mu(E) \inf_{x \in E} \left\{ \limsup_{r \rightarrow 0} \frac{r^s}{\mu(B(x, r))} \right\} \leq \mathcal{P}^s(E) \leq c_4 \mu(E) \sup_{x \in E} \left\{ \limsup_{r \rightarrow 0} \frac{r^s}{\mu(B(x, r))} \right\}. \quad (2.2)$$

where  $B(x, r)$  denotes the  $r$ -closed ball center at  $x$ .

Then, suppose that  $\nu$  is a finite Borel measure. We define  $\dim(\nu)$  and  $\text{Dim}(\nu)$  by<sup>[5]</sup>

$$\dim(\nu) = \inf_E \{ \dim(E) : \nu(S \setminus E) = 0 \}. \quad (2.3)$$

$$\text{Dim}(\nu) = \inf_E \{ \text{Dim}(E) : \nu(S \setminus E) = 0 \}. \quad (2.4)$$

Furthermore  $C^s(\nu)$  and  $\mathcal{P}^s(\nu)$  are defined as follows:

$$C^s(\nu) = \inf \{ C^s(Y) : \nu(S \setminus Y) = 0 \}, \quad \mathcal{P}^s(\nu) = \inf \{ \mathcal{P}^s(Y) : \nu(S \setminus Y) = 0 \}.$$

We shall call  $C^s(\nu)$  and  $\mathcal{P}^s(\nu)$  the  $s$ -dimensional centered Hausdorff measure and the  $s$ -dimensional packing measure of  $\nu$ , respectively. It is directly followed by the above definitions that

$$\dim(\nu) = \sup \{s \geq 0 : C^s(\nu) = \infty\} = \inf \{s \geq 0 : C^s(\nu) = 0\},$$

$$\text{Dim}(\nu) = \sup \{s \geq 0 : \mathcal{P}^s(\nu) = \infty\} = \inf \{s \geq 0 : \mathcal{P}^s(\nu) = 0\}.$$

**Definition 2.2** We define that two finite Borel measures  $\mu, \nu$  are equivalent on  $E$  and write  $\mu \sim \nu$  on  $E$  if the following condition holds:  $\mu(B) = 0$  if and only if  $\nu(B) = 0$  for any Borel set  $B \subset E$ .

### 3 The equivalence of measures on Moran set

Now we shall give some definitions and notations concerning symbolic dynamics and Moran set.

Let  $\{n_k\}_{k \geq 1}$  be a sequence of positive integers with  $n_k \geq 2, I_\infty = \{(i_1, i_2, \dots), 1 \leq i_j \leq n_j\}$ . For any  $k \in \mathbb{N}$ , let  $I_k = \{(i_1, i_2, \dots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}, I = \bigcup_{k \geq 0} I_k$ , where  $I_0 = \phi$ . The shift on  $I$  is denoted by  $\sigma: \sigma(i) = \sigma(i_1, i_2, i_3, \dots) = (i_2, i_3, i_4, \dots)$  for  $i = (i_1, i_2, i_3, \dots) \in I$ . In this paper, the space of  $\sigma$ -invariant ergodic Borel probability measures on  $I$  is denoted by  $\mathcal{E}_\sigma(I)$ .

**Definition 3.1**<sup>[4]</sup> Let  $(S, \rho)$  be a complete metric space, and suppose  $D \subset S$  a compact set with no empty interior(for convenience, we assume that the diameter of  $D$  is 1). The collection  $\mathcal{F} = \{D_i : i \in I\}$  of subsets of  $D$  is supposed to have Moran structure, if it satisfies:

(1) For any  $(i_1, i_2, \dots, i_k) \in I_k, D_{i_1 i_2 \dots i_k}$  is similar to  $D$ . That is, there exists a similar transformation  $S_{i_1 i_2 \dots i_k} : S \rightarrow S$  such that  $S_{i_1 i_2 \dots i_k}(D) = D_{i_1 i_2 \dots i_k}$ . Assume that  $D_\phi = D$ .

(2)  $\forall k \geq 1, (i_1, i_2, \dots, i_{k-1}) \in I_{k-1}, D_{i_1 i_2 \dots i_k} (i_k = 1, 2, \dots, n_k)$  are subsets of  $D_{i_1 i_2 \dots i_{k-1}}$ , and

$$D_{i_1 i_2 \dots i_{k-1} i_k}^o \cap D_{i_1 i_2 \dots i_{k-1} i'_k}^o = \phi \quad 1 \leq i_k < i'_k \leq n_k,$$

where  $A^o$  denotes the interior of  $A$ .

(3) For any  $k \geq 1$  and any  $(i_1 i_2 \dots i_k) \in I_k$ , we have

$$0 < a_{i_1 i_2 \dots i_k} = \frac{|D_{i_1 i_2 \dots i_k}|}{|D_{i_1 i_2 \dots i_{k-1}}|} < 1 \quad k \geq 2.$$

where  $|A|$  denotes the diameter of  $A$ .

Suppose that  $\mathcal{F}$  is a collection of subsets of  $D$  having Moran structure, we call  $E = \bigcap_{k \geq 1} \bigcup_{i \in I_k} D_i$  a Moran set determined by  $\mathcal{F}$ , and call  $\mathcal{F}_k = \{D_i : i \in I_k\}$  the  $k$ -order fundamental sets of  $E$ ,  $D$  is called the original set of  $E$ .

**Remark.** If  $\lim_{k \rightarrow \infty} \max_{i \in I_k} |D_i| > 0$ , then the corresponding Moran set  $E$  will contain interior points. Thus the measure and dimension properties will be trivial. Therefore

$$\lim_{k \rightarrow \infty} \max_{i \in I_k} |D_i| = 0. \tag{3.1}$$

Then the set  $\bigcap_{n=1}^{\infty} D_{i_1 \dots i_n}$  consists of a single point which is denoted by  $\varphi(i)$ :

$$\varphi(i) = \bigcap_{n=1}^{\infty} D_{i_1 \dots i_n}.$$

For convenience,  $i|_n$  denotes the first  $n$  elements of the sequence  $i_1 i_2 \dots i_n \dots$ . We will use the abbreviations,  $a_n(i) = a_{i|_n} = a_{i_1 i_2 \dots i_n}$  and  $D_n(i) = D_{i|_n} = D_{i_1 i_2 \dots i_n}$  for  $i \in I_\infty$ .

From the definition of  $a_n(i)$ , we know that  $0 < a_n(i) < 1, \forall i \in I$ . By the density of real numbers, there exists  $a \in (0, 1)$  such that

$$a_n(i) > a, \quad \text{for any } i \in I, \quad n \in N.$$

By the Definition 3.1 and the above inequality, we see that

$$a < a_n(i) < 1 - (n_1 - 1)a \quad \text{for any } i \in I. \tag{3.2}$$

Here we consider a Moran set  $E$ , which satisfies the strong separation condition:

For any  $\varphi(i) \in E$ , there exists  $0 < \delta_n \leq \frac{1}{2}$  such that

$$D_n(i') \cap B(\varphi(i), \delta_n |D_n(i)|) = \phi, \tag{3.3}$$

if  $i_k = i'_k, \quad k = 1, 2, \dots, n - 1, \quad i_n \neq i'_n$  and  $\delta = \inf_{n \geq 1} \delta_n > 0$ , where  $i' = (i'_1, i'_2, \dots, i'_n \dots)$ .

**Lemma 3.2**<sup>[4]</sup> Suppose that  $E \subset D$  is a Moran set satisfying (3.3). Let  $\mu$  be a finite Borel measure such that  $\text{supp}(\mu) \subseteq E$ , the following hold for any  $\varphi(i) \in E$ :

$$(a\delta)^s \liminf_{n \rightarrow \infty} \frac{|D_n(i)|^s}{\mu(D_n(i))} \leq \liminf_{r \rightarrow 0} \frac{r^s}{\mu(B(\varphi(i), r))} \leq a^{-s} \liminf_{n \rightarrow \infty} \frac{|D_n(i)|^s}{\mu(D_n(i))}, \tag{3.4}$$

$$(a\delta)^s \limsup_{n \rightarrow \infty} \frac{|D_n(i)|^s}{\mu(D_n(i))} \leq \limsup_{r \rightarrow 0} \frac{r^s}{\mu(B(\varphi(i), r))} \leq a^{-s} \limsup_{n \rightarrow \infty} \frac{|D_n(i)|^s}{\mu(D_n(i))}. \tag{3.5}$$

**Theorem 3.3**<sup>[4]</sup> Suppose that  $E \subset D$  is a Moran set satisfying (3.3). Let  $\mu$  be a finite Borel measure such that  $\text{supp}(\mu) \subseteq E$ .

(1) Suppose that there exists  $\alpha$  such that

$$\limsup_{n \rightarrow \infty} \frac{\mu(D_n(i))}{\left(\prod_{j=1}^n a_j(i)\right)^\gamma} = \begin{cases} 0 & \text{if } \gamma < \alpha, \\ \infty & \text{if } \gamma > \alpha, \end{cases} \text{ for any } i \in I_\infty.$$

Then  $\dim(E) = \alpha = \dim(\mu)$ .

(2) Suppose that  $\mu$  satisfies a stronger condition at  $\alpha$ :

$$0 < \limsup_{n \rightarrow \infty} \frac{\mu(D_n(i))}{\left(\prod_{j=1}^n a_j(i)\right)^\alpha} < \infty \text{ for any } i \in I_\infty.$$

Then  $\mu \sim \mathcal{C}^\alpha$  on  $E$ .

**Theorem 3.4**<sup>[4]</sup> Suppose that  $E \subset D$  is a Moran set satisfying (3.3). Let  $\nu$  be a finite Borel measure such that  $\text{supp}(\nu) \subseteq E$ .

(1) Suppose that there exists  $\beta$  such that

$$\liminf_{n \rightarrow \infty} \frac{\nu(D_n(i))}{\left(\prod_{j=1}^n a_j(i)\right)^\gamma} = \begin{cases} 0 & \text{if } \gamma < \beta, \\ \infty & \text{if } \gamma > \beta, \end{cases} \text{ for any } i \in I_\infty.$$

Then  $\text{Dim}(E) = \beta = \text{Dim}(\nu)$ .

(2) Suppose that  $\nu$  satisfies a stronger condition at  $\beta$ :

$$0 < \liminf_{n \rightarrow \infty} \frac{\nu(D_n(i))}{\left(\prod_{j=1}^n a_j(i)\right)^\beta} < \infty \text{ for any } i \in I_\infty.$$

Then  $\nu \sim \mathcal{P}^\beta$  on  $E$

## 4 The dimension of image measure

**Definition 4.1** Let  $\varphi : (\Omega, F) \rightarrow (\Omega', F')$  be a measurable map, where  $(\Omega, F)$  and  $(\Omega', F')$  are two measurable spaces. Let  $\mu$  be a measure on  $(\Omega, F)$ . Then we call image measure of  $\mu$  by  $\varphi$ , the measure on  $(\Omega', F')$ , denoted by  $\varphi^* \mu$  and defined by

$$\forall B \in F', \varphi^* \mu(B) = \mu(\varphi^{-1}(B)).$$

If the reference measures  $\mu$  and  $\nu$  in Theorems 3.3 and 3.4 are the images of  $\sigma$ -invariant ones in symbolic space  $I$ , certain results of invariant measures in ergodic theory are directly translated into the words of our geometrical measures. Here we put the following assumption,

$$0 < \liminf_{n \rightarrow \infty} \prod_{j=1}^n \frac{a_j(i)}{a_j(\sigma(i))} \leq \limsup_{n \rightarrow \infty} \prod_{j=1}^n \frac{a_j(i)}{a_j(\sigma(i))} < \infty \text{ for any } i \in I, \tag{4.1}$$

which ensures a self-similarity of the Moran sets in a weak sense. By virtue of (4.1), we can obtain similar formula of dimension of measures using entropy as in the case of self-similar sets (in Hutchinson's sense).

**Theorem 4.2** Let  $E \subset D$  be a Moran set satisfying (3.3).

(1) Suppose that  $\mu \in \mathcal{E}_\sigma(\mathcal{I})$  and  $\varphi^* \mu$  is the  $\varphi$ -image measure of  $\mu$ . Then

$$\dim(\varphi^* \mu) = h_\mu(\sigma) \left( \int \limsup_{n \rightarrow \infty} \left( -\frac{1}{n} \sum_{j=1}^n \log a_j(i) \right) d\mu \right)^{-1},$$

$$\text{Dim}(\varphi^* \mu) = h_\mu(\sigma) \left( \int \liminf_{n \rightarrow \infty} \left( -\frac{1}{n} \sum_{j=1}^n \log a_j(i) \right) d\mu \right)^{-1},$$

where  $h_\mu(\sigma)$  is the entropy of  $\sigma$  with respect to the measure  $\mu$ .

(2) Suppose that  $\mu \in \mathcal{E}_\sigma(\mathcal{I})$  and  $\alpha = \alpha(\mu) = \dim(\varphi^* \mu)$ . Then the following hold.

$$\mathcal{C}^\alpha(\varphi^* \mu) = 0 \iff \limsup_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\left( \prod_{j=1}^n a_j(i) \right)^\alpha} = \infty \quad \mu - a.e. \quad i.$$

$$0 < \mathcal{C}^\alpha(\varphi^* \mu) < \infty \iff 0 < \limsup_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\left( \prod_{j=1}^n a_j(i) \right)^\alpha} < \infty \quad \mu - a.e. \quad i.$$

$$\mathcal{C}^\alpha(\varphi^* \mu) = \infty \iff \limsup_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\left( \prod_{j=1}^n a_j(i) \right)^\alpha} = 0 \quad \mu - a.e. \quad i.$$

(3) Suppose that  $\mu \in \mathcal{E}_\sigma(\mathcal{I})$  and  $\beta = \beta(\mu) = \text{Dim}(\varphi^* \mu)$ . Then the following hold.

$$\mathcal{P}^\beta(\varphi^* \mu) = 0 \iff \liminf_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\left( \prod_{j=1}^n a_j(i) \right)^\beta} = \infty \quad \mu - a.e. \quad i.$$

$$0 < \mathcal{P}^\beta(\varphi^* \mu) < \infty \iff 0 < \liminf_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\left( \prod_{j=1}^n a_j(i) \right)^\beta} < \infty \quad \mu - a.e. \quad i.$$

$$\mathcal{P}^\beta(\varphi^* \mu) = \infty \iff \liminf_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\left( \prod_{j=1}^n a_j(i) \right)^\beta} = 0 \quad \mu - a.e. \quad i.$$

**Proof.**(1) First let us recall

$$|D_n(i)| = \prod_{j=1}^n a_j(i).$$

From (2.1) and (3.4), we can see that

$$d_1 \varphi^* \mu(Y) \inf_{i \in \varphi^{-1}(Y)} \left( \limsup_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\prod_{j=1}^n a_j(i)^\alpha} \right)^{-1} \leq \mathcal{C}^\alpha(Y) \leq d_2 \varphi^* \mu(Y) \sup_{i \in \varphi^{-1}(Y)} \left( \limsup_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\prod_{j=1}^n a_j(i)^\alpha} \right)^{-1}. \quad (4.2)$$

Set

$$A^+(i) = \limsup_{n \rightarrow \infty} \left( -\frac{1}{n} \sum_{j=1}^n \log a_j(i) \right).$$

Then  $A^+(i) \geq 0$ , and the condition (4.1) clearly implies that  $A^+(i) = A^+(\sigma(i))$ . Since  $\mu$  is ergodic, there exists  $A_\mu^+$  such that the set  $\{i \in I : A^+(i) = A_\mu^+\}$  has full measure. Set

$$I_\mu = \{i \in I : A^+(i) = A_\mu^+, \lim_{n \rightarrow \infty} \frac{1}{n} \log \mu((i_1, i_2, \dots, i_n)) = -h_\mu(\sigma)\}.$$

Then  $\mu(I_\mu) = 1$  by the Shannon-McMillan-Breiman Theorem. It is a routine work to compute  $\dim(I_\mu) = h_\mu(\sigma)/A_\mu^+$  if we use (4.2), which directly leads to  $\dim(I_\mu) \leq h_\mu(\sigma)/A_\mu^+$ .

On the other hand, if  $\mu(Y) = 1$ , then the set  $Y \cap I_\mu$  has full measure and its centered Hausdorff dimension is equal to  $h_\mu(\sigma)/A_\mu^+$  by the same reasoning. This shows that  $\dim(I_\mu) \geq h_\mu(\sigma)/A_\mu^+$  and together with the above upper estimate, we have  $\dim(I_\mu) = h_\mu(\sigma)/A_\mu^+$ . Finally, it is easy to see that

$$A_\mu^+ = \int \limsup_{n \rightarrow \infty} \left( -\frac{1}{n} \sum_{j=1}^n \log a_j(i) \right) d\mu$$

and the first equality is proved. Note that

$$-\log b \leq A^+(i) = \limsup_{n \rightarrow \infty} \left( -\frac{1}{n} \sum_{j=1}^n \log a_j(i) \right) \leq -\log a$$

so  $0 < A^+(i) < \infty$ . The second equality can be shown in just the same way.

(2) If we set  $B^+(i) = \limsup_{n \rightarrow \infty} \left( \varphi^* \mu(D_n(i)) \right) / \left( \prod_{j=1}^n a_j(i)^\alpha \right)$ , then (4.1) and the ergodicity state that the sets  $\{B^+ = 0\}$ ,  $\{0 < B^+ < \infty\}$  and  $\{B^+ = \infty\}$  are all either null or full with respect to the measure  $\mu$ .

Suppose that  $\mathcal{C}^\alpha(\varphi^* \mu) = 0$ . Then for any  $n \in N$  there exists  $Y_n \subset E$  such that  $\varphi^* \mu(E \setminus Y_n) = 0$  and  $\mathcal{C}^\alpha(Y_n) < \frac{1}{n}$ . Let  $B$  be one of the above sets  $\{B^+ = 0\}$ ,  $\{0 < B^+ < \infty\}$  and  $\{B^+ = \infty\}$  such that  $\mu(B) = 1$  and we set  $Z_{n,k} = Y_n \cap \varphi(B_k)$  where  $B_k = \{i \in I : \frac{1}{k} \leq B^+ \leq k\}$  if  $B = \{0 < B^+ < \infty\}$ ,  $B_k = \{i \in I : B^+ = 0\}$  if  $B = \{B^+ = 0\}$  and  $B_k = \{i \in I : B^+ = \infty\}$  if  $B = \{B^+ = \infty\}$ . Since  $\mu(B_k) \rightarrow 1$  as  $k \rightarrow \infty$ ,  $\lim_{k \rightarrow \infty} \varphi^* \mu(Z_{n,k}) = 1$  for any  $n \in N$ . Moreover

$$\mathcal{C}^\alpha(Z_{n,k}) \leq \mathcal{C}^\alpha(Y_n) < \frac{1}{n} \text{ for any } k \in N.$$

Then putting  $Y = Z_{n,k}$  in the left hand side of (4.2), we obtain

$$d_1 \varphi^* \mu(Z_{n,k}) \inf_{i \in \varphi^{-1}(Z_{n,k})} B^+(i)^{-1} \leq \mathcal{C}^\alpha(Z_{n,k}) < \frac{1}{n}.$$

Let  $k$  be large enough to satisfy  $\varphi^* \mu(Z_{n,k}) > 1/2$ , we can observe that

$$\inf_{i \in \varphi^{-1}(Z_{n,k})} B^+(i)^{-1} = \left( \sup_{i \in \varphi^{-1}(Z_{n,k})} B^+(i) \right)^{-1} \leq 2/(d_1 n),$$

which clearly creates a contradiction if  $B = \{B^+ = 0\}$ . On the other hand, if  $B = \{0 < B^+ < \infty\}$ , the above inequality implies that  $k^{-1} \leq 2(d_1 n)^{-1}$  for any  $n$  and any sufficiently large  $k$ . But this is clearly impossible. Therefore  $B = \{B^+ = \infty\}$  follows.

Suppose conversely that  $B^+ = \infty$   $\mu - a.e. i$ . Then the right hand side of (4.2) clearly implies that  $\mathcal{C}^\alpha(Y) = 0$  whenever  $\varphi^* \mu(Y) = 1$ . Therefore  $\mathcal{C}^\alpha(\varphi^* \mu) = 0$  by the definition and the first assertion is proved. The last assertion can be proved similarly and automatically the second assertion is true as well.

Similarly, (3) follows the estimate

$$d_3 \varphi^* \mu(Y) \inf_{i \in \varphi^{-1}(Y)} \left( \liminf_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\prod_{j=1}^n a_j(i)^\beta} \right)^{-1} \leq \mathcal{P}^\beta(Y) \leq d_4 \varphi^* \mu(Y) \sup_{i \in \varphi^{-1}(Y)} \left( \liminf_{n \rightarrow \infty} \frac{\varphi^* \mu(D_n(i))}{\prod_{j=1}^n a_j(i)^\beta} \right)^{-1}$$

which is obtained from (2.2) and (3.5).  $\square$

## 5 Statistical mechanical characterization for dimensions

In this section we consider simple cases in which the condition of all the numbers  $a_{i_1 \dots i_n}$  depend only on the length and the last variable:  $a_{i_1 \dots i_n} = a_{n, i_n}$ . In these cases the centered Hausdorff dimension and the packing dimension of  $E$  are exactly characterized through a statistical mechanical function and the centered

Hausdorff and the packing measures both coincide with the Gibbs measure up to constants. These are simple analogies of the characterization of the dimensions of repellers through so-called thermodynamic formalism<sup>[9]</sup>.

We firstly define the lower pressure function  $\underline{\pi}(\cdot)$  and the upper pressure function  $\overline{\pi}(\cdot)$  by

$$\underline{\pi}(\gamma) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\gamma \right), \quad (5.1)$$

$$\overline{\pi}(\gamma) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\gamma \right). \quad (5.2)$$

Secondly we define  $\gamma$ -Gibbs measure  $\mu_\gamma$  by

$$\mu_\gamma((i_1 i_2 \dots i_n)) = \prod_{j=1}^n \frac{a_{j,i_j}^\gamma}{Z_n^\gamma}, \quad Z_n^\gamma = \sum_j a_{n,j}^\gamma \quad \text{for any } (i_1 i_2 \dots i_n) \in I_n, n \in \mathbb{N}. \quad (5.3)$$

It is clear that the measure  $\mu_\gamma$  uniquely exists by Kolmogorov's extension theorem.

**Lemma 5.1** Both  $\underline{\pi}(\gamma)$  and  $\overline{\pi}(\gamma)$  are strictly decreasing and continuous in  $\gamma$ .

**Proof.** Suppose that  $\gamma' > \gamma$ . Then from (3.2) we have

$$\begin{aligned} \underline{\pi}(\gamma') &= \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^{\gamma'} \right) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\gamma a_{j,i_j}^{\gamma' - \gamma} \right) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\gamma b^{\gamma' - \gamma} \right) = (\gamma' - \gamma) \log b + \underline{\pi}(\gamma) \end{aligned}$$

and analogously

$$\underline{\pi}(\gamma') \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\gamma a^{\gamma' - \gamma} \right) = (\gamma' - \gamma) \log a + \underline{\pi}(\gamma).$$

These estimates clearly indicate the required properties of  $\underline{\pi}$ . Analogously we obtain the strict decrease and the continuity of the function  $\overline{\pi}$ .

Since  $\underline{\pi}(0) = \overline{\pi}(0) = \log n_*$  and  $\underline{\pi}(-\log n^*/\log b) \leq 0$ ,  $\overline{\pi}(-\log n^*/\log b) \leq 0$  for  $n_* = \inf_{k \geq 1} \{n_k\}$  and  $n^* = \sup_{k \geq 1} \{n_k\}$  by the above estimates, there exists a unique pair  $(\alpha, \beta)$  such that  $\underline{\pi}(\alpha) = 0$  and  $\overline{\pi}(\beta) = 0$ .  $\square$

**Theorem 5.2** The numbers  $\alpha = \dim(E)$  and  $\beta = \text{Dim}(E)$  are uniquely characterized by the following equations,

$$\underline{\pi}(\alpha) = 0, \quad \overline{\pi}(\beta) = 0.$$

Furthermore  $\mathcal{C}^\alpha \sim \varphi^* \mu_\alpha$  on  $E$  if

$$0 < \limsup_{n \rightarrow \infty} \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\alpha < \infty$$

and  $\mathcal{P}^\beta \sim \varphi^* \mu_\beta$  on  $E$  if

$$0 < \liminf_{n \rightarrow \infty} \sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\beta < \infty.$$

**Proof.** It immediately follows the Definition 4.1 and (5.3) that

$$\mu_\alpha((i_1 i_2 \dots i_n)) = \varphi^* \mu_\alpha(D_n(i)) = |D_n(i)|^\alpha \left( \prod_{j=1}^n Z_j^\alpha \right)^{-1}.$$

Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\varphi^* \mu_\alpha(D_n(i))}{|D_n(i)|^\gamma} &= \limsup_{n \rightarrow \infty} |D_n(i)|^{\alpha-\gamma} \left( \prod_{j=1}^n Z_j^\alpha \right)^{-1} \\ &= \lim_{n \rightarrow \infty} |D_n(i)|^{\alpha-\gamma} \exp\left(-\liminf_{n \rightarrow \infty} \sum_{j=1}^n \log Z_j^\alpha\right). \end{aligned}$$

Since clearly

$$\sum_{(i_1 i_2 \dots i_n) \in I_n} \prod_{j=1}^n a_{j,i_j}^\alpha = \prod_{j=1}^n Z_j^\alpha,$$

we have

$$\underline{\pi}(\alpha) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log Z_j^\alpha.$$

Henceforce in view of  $\underline{\pi}(\alpha) = 0$  we easily have

$$\limsup_{n \rightarrow \infty} \frac{\varphi^* \mu_\alpha(D_n(i))}{|D_n(i)|^\gamma} = \infty \quad \text{if } \gamma > \alpha$$

and

$$\limsup_{n \rightarrow \infty} \frac{\varphi^* \mu_\alpha(D_n(i))}{|D_n(i)|^\gamma} = 0 \quad \text{if } \gamma < \alpha.$$

Therefore Theorem 3.3 says that  $\dim(E) = \alpha = \dim(\varphi^* \mu_\alpha)$ . Analogously we obtain  $\text{Dim}(E) = \beta = \text{Dim}(\varphi^* \mu_\beta)$ . The latter statement easily follows Theorems 3.3 and 3.4.  $\square$

## 6 Conclusion

For the condition that the Moran sets satisfy the strong separation, we first recall the equivalence of the centered Hausdorff measure and the packing measure. From Theorem 4.2, we obtained similar formulae of dimension of measures using entropy if the measures  $\mu$  and  $\nu$  in Theorems 4.2 and 4.3 are the images of  $\sigma$ -invariant ones in symbolic space  $I$ . Finally, we treat a class of Moran sets and give some statistical interpretation on dimensions and corresponding geometrical measures.

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