ACADIMIC
World Academic Union

ISSN 1479-3889 (print), 1479-3897 (online)
International Journal of Nonlinear Science
Vol.3(2007) No.1,pp.27-31

A Numerical Algorithm for Finding Positive Solutions of Superlinear Dirichlet Problem

G.A.Afrouzi [1], Z.Naghizadeh, S.Mahdavi
Department of Mathematics, Faculty of Basic Sciences,
Mazandaran University, Babolsar, Iran
(Received 17 December 2005, accepted 9 July 2006)

Abstract: In this work we present a numerical approach for finding positive solutions for $-\Delta u = \lambda(u + u^2 + u^3)$ for $x \in \Omega$ with Dirichlet boundary condition. We will show that in which range of λ, this problem achieves a numerical solution and what is the behavior of the branch of this solutions.

Key words: Superlinear problem; Nehari manifold; positive solution; variational method

1 Introduction

The subject of partial differential equations, PDE, has many applications in real life. In particular nonlinear elliptic PDE have been used in many physical problems such as fluid dynamics, chemical reactions, and steady state solutions of reaction diffusion equations (see[5]).
When studying a nonlinear PDE, one might be interested in finding solutions which satisfy some boundary value problem, BVP.
It is essential to approximate the solution of partial differential equations numerically in order to investigate the predictions of the mathematical models, as the exact solutions are usually unavailable.
In this paper we want to obtain the positive solution of elliptic boundary value problem has the form

$$\begin{cases} -\Delta u(x) = \lambda(u + u^2 + u^3)(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \qquad (1)$$

where Δ is the standard Laplace operator, and Ω is a bounded domain in $R^N (N \leq 4)$ with smooth boundary. Similar problems are studied by using finite difference method[2].
We use a variational method to prove the existence of positive solutions. In order to do this we will define an action functional J on H. Let H be the Sobolev space $H = H_0^{1,2}(\Omega)$ with inner product $\langle u, v\rangle = \int_\Omega \nabla u.\nabla v dx$ (see[1]), and $J(u) = \int_\Omega \{\frac{1}{2}|\nabla u|^2 - \lambda F(u)\}dx$ where $F(u) = \int_0^u f(t)dt$. We consider in this paper $f(u) = \lambda(u + u^2 + u^3)$.
Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 < ...$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω.

2 Variational characterization

$f \in C^1(R, R)$ imply that $J \in C^2(H, R)$ and

[1]**Corresponding author**. *E-mail address*: afrouzi@umz.ac.ir

IJNS.2007.01.15/056

$$J'(u)(v) = \langle \nabla J(u), v \rangle = \int_\Omega (\nabla u.\nabla v - f(u)v)dx$$

Define $\gamma(u) = \langle \nabla J(u), u \rangle = \int_\Omega \{|\nabla u|^2 - uf(u)\}dx$ and compute

$$\gamma'(u)(v) = \langle \nabla \gamma(u), v \rangle = 2\int_\Omega (\nabla u.\nabla v - f(u)v - f'(u)uv)dx.$$

Definition 1 : For $u \in L^1(\Omega)$ we define $u_+(x) = \max\{u(x), 0\} \in L^1(\Omega)$ and $u_-(x) = \min\{u(x), 0\} \in L^1(\Omega)$ if $u \in H$ then $u_+, u_- \in H$ (see [4]). For $u \neq 0$ we say u is positive(and write $u > 0$) if $u_- = 0$, and similarly, u is negative ($u < 0$) if $u_+ = 0$.

Lemma 1 [3]: The function $h : H \to H$ defined by $h(u) = u_+$ is continuous.Also, h defines a continuous function from L^{p+1} into itself.

When J is bounded below on H, J has a minimizer on H which is a critical point of J. In many cases such as (1) J is not bounded below on H, but is bounded below on an appropriate subset of H and a minimizer on this set (if it exists) may give rise to solution of the corresponding differential equation. A good condidate for an appropriate subset of H is the Nehari submanifold of H defined :

$$S = \{u \in H - \{0\} : \gamma(u) = 0\},$$

where we note that nontrivial solution to (1) are in S, and S is a closed subset of H.

Lemma 2 [3]: Under the above assumption we have

a) 0 is a local minimum of J. If $u \in H - \{0\}$ then there exists a unique $\bar{t} = \bar{t}(u) \in (0, \infty)$ such that $\bar{t}u \in S$. Moreover $J(\bar{t}u) = \max_{t>0} J(tu) > 0$. If $\gamma(u) < 0$ then $\bar{t} < 1$, and if $\gamma(u) > 0$ then $\bar{t} > 1$ and $J(u) > 0$.
b) $u \in S$ is a critical point of J on H iff u is a critical point of $J|_S$.
c) $J|_S$ is coercive, i.e., $J(u) \to \infty$ as $\|u\| \to \infty$. Also $0 \notin S$ and $\inf J|_S > 0$.

Lemma 3 [3]: If $w \in S$, $w > 0$ and $J(w) = \min_{\{u \in S: u > 0\}} J$, then w is a critical point of J .

We can prove the **existence of positive solutions** in this way : If $f'(0) < \lambda_1$, by using the Sobolev imbedding theorem, the coercivity of J, and properties of γ, we can show that the minimizer of J from lemma 3 exists, and hence the solution $w > 0$ exists.

3 Numerical algorithm

Given a nonzero element $u \in H$ and a piece-wise smooth region $\Omega \subset R^N$, we will use the notation **u** to represent an array of real numbers agreeing with u on a grid $\Omega \subset \bar{\Omega}$. We will take the grid to be regular.

At each step of iterative process, we are required to project nonzero elements of H onto the submanifold S. By Lemma (2), we see that the projection of $\nabla J(u)$ on to the ray $\{\lambda u : \lambda > 0\}$ is given by

$$P_{\mathbf{u}}(J(u)) = \frac{\langle \nabla u, u \rangle}{\langle u, u \rangle} u = \frac{\gamma(u)}{\|u\|^2} u.$$

Let u be a nonzero element of H, represented by **u** over the grid Ω. Let $s_1 = 0.5$ or another perhaps optimally determined small positive constant. Define $\mathbf{u}_0 = \mathbf{u}$ and

$$\mathbf{u}_{k+1} = \mathbf{u}_k + s_1 \frac{\gamma(\mathbf{u}_k)}{\|\mathbf{u}_k\|^2} \mathbf{u}_k \qquad k \geq 0$$

We will use notation $P_1(\mathbf{u}) = \lim \mathbf{u}_k$, provided that limit exists, to represent unique positive multiple of u lying on S.

We use following algorithm to find the solution:

1. Initialize $\mathbf{u}_0$ with appropriate initial guess.

2. Project **u** on to S.(ray projection in Ascent direction element **u** onto S, that we explain it above).

The standard L^2 gradient is not the gradient we are considering,

$$\langle \nabla J(u), v\rangle = J'(u)(v) = \int_\Omega (\nabla u.\nabla v - f(u)v)$$

$$= \int_\Omega (\nabla u.\nabla v - (-\Delta)(-\Delta)^{-1}(f(u))v)dx$$

$$= \int_\Omega \{(\nabla u.\nabla v + \nabla(-\Delta)^{-1}(f(u)).\nabla v)\}dx$$

$$= \langle u - (-\Delta)^{-1}(f(u)), v\rangle.$$

3. Begin loop with $k = 0$

3.1. Solve linear system $-\Delta g = f(\mathbf{u}_k)$ for g allows one to explicitly construct the array $\nabla J(\mathbf{u}_k) \equiv \mathbf{u}_k - g$, representing $\nabla J(u)$.

3.2. Take gradient descent

$$\mathbf{u}_k = \mathbf{u}_k - s_2 \nabla J(\mathbf{u}_k)$$

3.3. Reproject $\mathbf{u}_k$ on to S.

3.4 . Increment k and repeat step (3.1),(3,2),(3,3) until convergence criteria are met:

$$\|\nabla J(\mathbf{u}_k)\|_2^2 \approx 0,\ \|\Delta \mathbf{u}_k + f(\mathbf{u}_k)\|_2^2 \approx 0.$$

In the following sections we look for the solutions , the existence of which is mentioned in introduction.

4 Numerical result : the ODE case

Now we apply the algorithm for our bvp,

$$\begin{cases} -\Delta u(x) = \lambda(u + u^2 + u^3)(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

Let $\Omega = [0, 1]$, initial function $u_0 = \frac{1}{10} sin(\pi x)$.
The algorithm presented in previous section depend on choice of grid and stability is confirmed by allowing the program to execute for more iterations.

By considering the well known bifurcation diagram, we would expect, due to increasing values of λ, the values of $\|u\|_\infty$ decreasing(see table **I**). Also we would expect $\|u\| \to \infty$ as $\lambda \to 0$ (we execute our MATLAB program for eps= 2.22×10^{-16}) and $\|u\| \to 0$ as $\lambda \to \lambda_1$.(see table **II**)

Table **I:** $\| u \|_\infty$ increases with respect to λ

grid	λ	$\|\nabla J(u)\|$	$\|u\|_\infty$
n=50	1	3.09×10^{-3}	4.26×10^{-3}
n=50	3	1.24×10^{-3}	2.09×10^{-3}
n=50	5	6.6×10^{-4}	1.47×10^{-3}
n=50	7	2.08×10^{-4}	9.2×10^{-4}

Table **II:** $\| u \|_\infty \to 0$ as $\lambda \to \lambda_1$

grid	λ	$\|\nabla J(u)\|$	$\|u\|_\infty$
n=50	eps	2.29×10^7	3.32×10^7
n=50	10	4×10^{-5}	$7.4 \times 1o^{-4}$

So according to the obtained results in the above table we guess that λ_1 is between 9 and 10.

5 Numerical result : the PDE case

Now we apply the algorithm for $-\Delta u(x) = \lambda(u + u^2 + u^3)(x)$, $x \in \Omega$ where $\Omega = [0,1] \times [0,1]$. Using initial function $u_0 = sin(\pi x)sin(\pi y)$.

As ODE case, due to increasing values of λ, the values of $\|u\|_\infty$ decreasing(represented in table **III**).

Table **III:** $\| u \|_\infty$ increases with respect to λ

grid	its	λ	$\|\nabla J(u)\|$	$\|\Delta u + f(u)\|$	$\|u\|_\infty$
n=50	7	1	0.137	14.658	5.330
n=50	7	5	0.032	7.853	1.777
n=50	7	9	0.012	4.851	0.966
n=50	1	25	0.002	0.014	0.003

Also we would expect $\|u\| \to \infty$ as $\lambda \to 0$ and $\|u\| \to 0$ as $\lambda \to \lambda_1$.(table **IV**)

Table **IV:** $\| u \|_\infty \to 0$ as $\lambda \to \lambda_1$

grid	λ	$\|u\|_\infty$
n=50	eps	3.98×10^8
n=50	25	0.003

So according to the obtained results in the above table we guess that λ_1 is between 20 and 30.

We would expect that Increased accuracy and efficiency would be gained by using more divisions and allowing the program to execute for more iterations.Table**V** certifies this.

Table **V:** The efficiency of n and its on $\| \nabla J(u) \|$

grid	λ	its	$\|\nabla J(u)\|$
n=30	10	3	0.013
n=30	10	7	0.009
n=50	10	3	0.012
n=50	10	7	0.008

Let $\lambda = 10 < \lambda_1, n = 20$. We approximate positive solution to the PDE(table **VI**)

Table **VI:** Approximate solution for $\lambda = 10$

	3	5	7	9	11
3	0.335	0.637	0.878	0.1033	0.1086
5	0.637	0.1284	0.1672	0.1968	0.2070
7	0.878	0.1672	0.2305	0.2713	0.2853
9	0.1033	0.1968	0.2713	0.3193	0.3359
11	0.1086	0.2070	0.2853	0.3359	0.3533

References

[1] R. A.Adams: Sobolev spaces. *Pure and Applied Mathematics*. Vol. 65, Academic Press, New York. (1975)

[2] G. A.Afrouzi and S.Khademloo: A numerical method to find positive solution of semilinear elliptic Dirichlet problems. *Applied Mathematics and Computation.* Article in Press.(2005)

[3] A. Castro, Jorge Cossio, and John M. Neuberger:Sighn changing Solution for a Superlinear Dirichlet Problem. Preprint, Accepted Rockey Mountain J. M. (1995)

[4] D. kinderlehrer and G. Stampacchia. Introduction to variational inequalities and their applications. *Academic Press, New York.* (1979)

[5] J. Smoller:Shock waves and reaction-diffusion equation.*Springer-Verlag* (second edition). (1994)