

# Smoothness Analysis of Coalescence Hidden Variable Fractal Interpolation Functions

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**Abstract.** The smoothness analysis of coalescence hidden variable fractal interpolation functions (CHFIFs) has been carried out in the present paper by using the operator approximation technique. It is found that the deterministic construction of functions having order of modulus continuity  $O(|t|^\delta(\log |t|)^m)$  is possible through CHFIFs, where  $m$  is a non-negative integer and  $0 < \delta \leq 1$ . The bounds of fractal dimension of CHFIFs are also obtained in critical cases.

**Keywords:** iterated function systems; attractor; fractal interpolation functions; hidden variable; self-affine; non-self-affine; operator approximation; smoothness analysis.

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## 1 Introduction

The fractal curves are often observed to occur in various disciplines such as natural sciences [1–3], engineering applications [4–7], economics [8] etc. These curves are constructed deterministically [9, 10] by using fractal interpolating function (FIF) from suitable choice of iterated function system (IFS). FIFs are generally self-affine in nature and the Hausdorff-Besicovitch dimensions of their graphs are non-integers. To approximate non-self-affine patterns, hidden variable FIFs (HFIFs) are constructed in [10–12] by projecting vector valued FIF corresponding to a generalized interpolation data. However, in practical applications of FIF, the interpolation data might be generated simultaneously from self-affine or non-self-affine functions [13], needing the construction of hidden variable bivariate fractal interpolation surfaces [14] by using *constrained free variables*. The concept of coalescence hidden variable FIF (CHFIF) that is self-affine or non-self-affine depending on the parameters of defining IFS is introduced in [15, 16].

Since FIFs are continuous but generally nowhere differentiable functions, their analysis can not be done satisfactorily by restricting to classical analysis tools. For the applications of FIF theory, in general, an expansion of the FIF in terms of a suitable function system is usually considered. Barnsley and Harrington [17] used shifted composition to express affine FIFs and computed their fractal dimensions. However, a practical implementation of this representation is somewhat difficult. Zhen [18] proposed another series representation of self-affine FIF through a new function  $\psi_{\sigma\kappa\omega}$  to study the Hölder property of FIF. Since, the function  $\psi_{\sigma\kappa\omega}$  has too many points of discontinuity, it is slightly tedious to analyze it in applications. Zhen and Gang [19] expanded equidistant FIF on  $[0, 1]$  by using Haar-wavelet function system and obtained their global Hölder property, when the number of interpolation points is  $N = 2^p + 1$ ,  $p$  being a definite positive integer. Gang [20] employed the technique of operator approximation to characterize the Hölder continuity

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of self-affine FIFs on a general set of nodes on  $[0, 1]$ . Bedford [21] obtained the Hölder exponent  $h$  of a self-affine fractal function that has non-linear scaling, using code space of  $n$  symbols associated with the IFS. He also showed the existence of a larger Hölder exponent  $h_\lambda$  defined at almost every point with respect to Lebesgue measure. The Hölder exponent needed in smoothness analysis of HFIFs is not yet studied due to interdependence of the components in the construction of HFIFs.

In the present paper, the approximation of a CHFIF is obtained through an operator found with integral averages on each subinterval of the CHFIF. Using this approximation, the Hölder exponent of the non-self-affine functions arising from IFS is found for the first time. The bounds of Fractal dimension of the CHFIF in critical cases are obtained. The organization of the paper is as follows: In Section 2, we give basic theory of CHFIFs. The Hölder continuity of CHFIFs is investigated in Section 3 by using the operator approximation technique. The bounds on fractal dimension of CHFIFs in critical cases are obtained in Section 4.

## 2 Theory of CHFIF

Let the interpolation data be  $\{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, 2, \dots, N\}$ , where  $-\infty < x_0 < x_1 < \dots < x_N < \infty$ . For constructing an interpolation function  $f_1 : [x_0, x_N] \rightarrow \mathbb{R}$  such that  $f_1(x_i) = y_i$  for all  $i = 0, 1, 2, \dots, N$ , consider a generalized set of data  $\{(x_i, y_i, z_i) \in \mathbb{R}^3 | i = 0, 1, 2, \dots, N\}$ , where  $z_i, i = 0, 1, 2, \dots, N$  are real parameters. Denote  $I = [x_0, x_N]$ ,  $I_i = [x_{i-1}, x_i]$ ,  $\tilde{g}_1 = \underset{i}{Min} y_i$ ,  $\tilde{g}_2 = \underset{i}{Max} y_i$ ,  $\tilde{h}_1 = \underset{i}{Min} z_i$ ,  $\tilde{h}_2 = \underset{i}{Max} z_i$  and  $K = I \times D$ , where  $D = J_1 \times J_2$ ,  $J_1, J_2$  are suitable compact sets in  $\mathbb{R}$  such that  $[\tilde{g}_1, \tilde{g}_2] \times [\tilde{h}_1, \tilde{h}_2] \subset D$ . Let  $L_i : I \rightarrow I_i$  be a contractive homeomorphism and  $F_i : K \rightarrow D$  be a continuous vector valued function such that for  $i = 1, 2, \dots, N$ ,

$$\left. \begin{aligned} L_i(x_0) = x_{i-1}, L_i(x_N) = x_i, \\ F_i(x_0, y_0, z_0) = (y_{i-1}, z_{i-1}), F_i(x_N, y_N, z_N) = (y_i, z_i), \end{aligned} \right\} \quad (2.1)$$

and for  $(x, y, z), (x^*, y, z), (x, y^*, z^*) \in K$ , for all  $i = 1, 2, \dots, N$ ,

$$\left. \begin{aligned} d(F_i(x, y, z), F_i(x^*, y, z)) &\leq c |x - x^*|, \\ d(F_i(x, y, z), F_i(x, y^*, z^*)) &\leq s d_E((y, z), (y^*, z^*)), \end{aligned} \right\} \quad (2.2)$$

where  $c$  and  $s$  are positive constants with  $0 \leq s < 1$ ,  $d$  is the sup. metric on  $K$ , and  $d_E$  is the Euclidean metric on  $\mathbb{R}^2$ . In the construction of CHFIF, choose  $L_i(x) = a_i x + b_i$  and  $F_i(x, y, z) = A_i(y, z)^T + (p_i(x), q_i(x))^T$ , where  $A_i$  is an upper triangular matrix  $\begin{pmatrix} \alpha_i & \beta_i \\ 0 & \gamma_i \end{pmatrix}$  and  $p_i(x), q_i(x)$  are continuous functions having two free parameters. We take  $\alpha_i$  as free variable with  $|\alpha_i| < 1$  and  $\beta_i$  as *constrained free variable* with respect to  $\gamma_i$  such that  $|\beta_i| + |\gamma_i| < 1$ . The generalized IFS that is needed for construction of CHFIF corresponding to the data  $\{(x_i, y_i, z_i) | i = 0, 1, \dots, N\}$  is defined as

$$\{\mathbb{R}^3; \omega_i(x, y, z) = (L_i(x), F_i(x, y, z)), i = 1, 2, \dots, N\}. \quad (2.3)$$

It is known [11] that the IFS defined in (2.3) associated with the data  $\{(x_i, y_i, z_i), i = 0, 1, \dots, N\}$  is hyperbolic with respect to a metric  $d^*$  on  $\mathbb{R}^3$  equivalent to the Euclidean metric. In particular, there exists a unique nonempty compact set  $G \subseteq \mathbb{R}^3$  such that  $G = \bigcup_{i=1}^N \omega_i(G)$ . The following proposition gives the existence of a unique vector valued function  $f$  that interpolates the generalized interpolation data and also establishes that the graph of  $f$  equals the attractor  $G$  of the generalized IFS:

**Proposition 2.1.** [11] *The attractor  $G$  of the IFS defined in (2.3) is the graph of the continuous vector valued function  $f : I \rightarrow D$  such that  $f(x_i) = (y_i, z_i)$  for all  $i = 1, 2, \dots, N$ , i.e.  $G = \{(x, y, z) : x \in I \text{ and } f(x) = (y(x), z(x))\}$ .*

The vector valued function  $f$  is fixed point of Read-Bajraktarević operator  $T$  that is defined on the metric space  $(\mathcal{F}, \rho)$ , where  $\mathcal{F} = \{g : I \rightarrow D | g \text{ is continuous, } g(x_0) = (y_0, z_0), g(x_N) = (y_N, z_N)\}$ , for

$g, h \in \mathcal{F}$ ,  $\rho(g, h) = \sup_{x \in I} \|g(x) - h(x)\|$ , and  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^2$ . Since  $f$  is the fixed point of  $T$ , we have for  $x \in I_i$ ,

$$f(x) = (Tf)(x) = F_i(L_i^{-1}(x), y(L_i^{-1}(x)), z(L_i^{-1}(x))), \quad i = 1, 2, \dots, N. \quad (2.4)$$

Let the vector valued function  $f : I \rightarrow D$  be written as  $f(x) = (f_1(x), f_2(x))$ . The required CHFIF is now defined as follows:

**Definition 2.1.** Let  $\{(x, f_1(x)) : x \in I\}$  be the projection of the attractor  $G$  on  $\mathbb{R}^2$ . Then, the function  $f_1(x)$  is called coalescence hidden variable FIF (CHFIF) for the given interpolation data  $\{(x_i, y_i) \mid i = 0, 1, \dots, N\}$ .

The following definition is needed to characterize CHFIF:

**Definition 2.2.** Let  $S$  be a set of points  $x = (x_1, x_2, \dots, x_E)$  in an Euclidean space of dimension  $E$  and  $rS = \{(r_1x_1, r_2x_2, \dots, r_Ex_E) : r = (r_1, r_2, \dots, r_E), r_n > 0 \text{ and } x \in S\}$ . The set  $S$  is called self-affine, if  $S$  is the union of  $N$  distinct subsets, each identical with  $rS$  up to translation and rotation. If  $S$  is not self-affine, then it is called non-self-affine. If  $r_1 = r_2 = \dots = r_E$ , then a self-affine set is called self-similar and a non-self-affine set is called non-self-similar.

**Remark 2.1.** Although, the attractor  $G$  of the IFS defined in (2.3) is a union of affine transformations of itself, the projection of the attractor is not always union of affine transformations of itself. Hence, CHFIFs are generally non-self-affine in nature. By choosing  $y_i = z_i$  and  $\alpha_i + \beta_i = \gamma_i$ , CHFIF  $f_1(x)$  coincides with a self-affine fractal function  $f_2(x)$  for the same interpolation data. Hence, the CHFIF is self-affine or self-similar in this case even though the class of CHFIFs is a subclass of the class of HFIFs. These CHFIFs can be used to approximate the random steps of Gaussian, increments of the fractional Brownian function and wave-height functions [2].

### 3 Smoothness analysis of CHFIF

In this section, the smoothness of CHFIFs is studied by using their operator approximations. The Hölder exponent of CHFIFs is calculated in the proof of our main Theorems 3.1-3.3.

We take the interpolation data on x-axis as  $0 = x_0 < x_1 < \dots < x_N = 1$ . Let the function  $F_i$  of the IFS (2.3) be of the form

$$F_i(x, y, z) = (\alpha_i y + \beta_i z + p_i(x), \gamma_i z + q_i(x)), \quad (3.1)$$

where  $|\alpha_i| < 1$ ,  $|\beta_i| + |\gamma_i| < 1$ ,  $p_i \in Lip\lambda_i$  ( $0 < \lambda_i \leq 1$ ), and  $q_i \in Lip\mu_i$  ( $0 < \mu_i \leq 1$ ). From (2.4) and (3.1), for  $x \in I_i$ , the fixed point  $f$  of  $T$  satisfies

$$\begin{aligned} Tf(x) &= F_i(L_i^{-1}(x), f_1(L_i^{-1}(x)), f_2(L_i^{-1}(x))) \\ \Rightarrow f(x) &= F_i(L_i^{-1}(x), f_1(L_i^{-1}(x)), f_2(L_i^{-1}(x))) \\ \Rightarrow (f_1(x), f_2(x)) &= (\alpha_i f_1(L_i^{-1}(x)) + \beta_i f_2(L_i^{-1}(x)) + p_i(L_i^{-1}(x)), \gamma_i f_2(L_i^{-1}(x)) + q_i(L_i^{-1}(x))). \end{aligned}$$

Following Definition 2.1, the CHFIF in this case can be written as

$$f_1(L_i(x)) = \alpha_i f_1(x) + \beta_i f_2(x) + p_i(x), \quad x \in I \quad (3.2)$$

where the self-affine fractal function  $f_2(x)$  is given by

$$f_2(L_i(x)) = \gamma_i f_2(x) + q_i(x), \quad x \in I. \quad (3.3)$$

Let  $I_{r_1} = [x_{r_1-1}, x_{r_1}] = L_{r_1}(I)$ . Then,  $I_{r_1} = L_{r_1}(0) + |I_{r_1}|I$ , where  $|x_{r_1} - x_{r_1-1}|$  is the length of  $I_{r_1}$ ,  $1 \leq r_1 \leq N$ . Similarly,  $I_{r_1 r_2} = L_{r_2}(0) + |I_{r_2}|L_{r_1}(I) = L_{r_2} \circ L_{r_1}(I) = L_{r_1 r_2}(I)$ , where  $|I_{r_1 r_2}| = |I_{r_1}| \cdot |I_{r_2}|$ ,  $1 \leq r_1, r_2 \leq N$ . In general,

$$I_{r_1 r_2 \dots r_m} = L_{r_m}(0) + |I_{r_m}|I_{r_1 r_2 \dots r_{m-1}} = L_{r_m} \circ L_{r_{m-1}} \circ \dots \circ L_{r_1}(I) = L_{r_1 r_2 \dots r_m}, \quad (3.4)$$

where  $|I_{r_1 r_2 \dots r_m}| = |I_{r_1}| \cdot |I_{r_2}| \dots |I_{r_m}|$  and  $1 \leq r_1, r_2, \dots, r_m \leq N$ .

We need the following lemmas for our main results:

**Lemma 3.1.** Let  $f_1$  be defined as in (3.2) and  $b_{r_1 r_2 \dots r_m} = \int_{I_{r_1 r_2 \dots r_m}} f_1(x) dx$ . Then,

$$b_{r_1 r_2 \dots r_m} = \sum_{k=1}^m \prod_{j=k+1}^m (|I_{r_j}| \alpha_{r_j}) |I_{r_k}| \left( \int_{I_{r_1 r_2 \dots r_{k-1}}} p_{r_k}(\xi) d\xi + \beta_{r_k} a_{r_1 r_2 \dots r_{k-1}} \right) + \prod_{j=1}^m (|I_{r_j}| \alpha_{r_j}) \int_0^1 f_1(\xi) d\xi, \quad (3.5)$$

where  $I_{r_0} = I$  and  $a_{r_1 r_2 \dots r_m} = \int_{I_{r_1 r_2 \dots r_m}} f_2(x) dx$ .

*Proof.* Since,  $b_{r_1 r_2 \dots r_m} = \int_{L_{r_m}(0) + |I_{r_m}| I_{r_1 r_2 \dots r_{m-1}}} f_1(x) dx$ , a change of the variable  $x$  by  $x = L_{r_m}(0) + |I_{r_m}| \xi$  gives

$$\begin{aligned} b_{r_1 r_2 \dots r_m} &= \int_{I_{r_1 r_2 \dots r_{m-1}}} f_1(L_{r_m}(0) + |I_{r_m}| \xi) |I_{r_m}| d\xi = |I_{r_m}| \int_{I_{r_1 r_2 \dots r_{m-1}}} f_1(L_{r_m}(\xi)) d\xi \\ &= |I_{r_m}| \int_{I_{r_1 r_2 \dots r_{m-1}}} (\alpha_{r_m} f_1(\xi) + \beta_{r_m} f_2(\xi) + p_{r_m}(\xi)) d\xi \\ &= |I_{r_m}| \left[ \int_{I_{r_1 r_2 \dots r_{m-1}}} p_{r_m}(\xi) d\xi + \beta_{r_m} \int_{I_{r_1 r_2 \dots r_{m-1}}} f_2(\xi) d\xi + \alpha_{r_m} \int_{I_{r_1 r_2 \dots r_{m-1}}} f_1(\xi) d\xi \right] \\ &= |I_{r_m}| \left[ \int_{I_{r_1 r_2 \dots r_{m-1}}} p_{r_m}(\xi) d\xi + \beta_{r_m} a_{r_1 r_2 \dots r_{m-1}} \right] + |I_{r_m}| |I_{r_{m-1}}| \alpha_{r_m} \left[ \int_{I_{r_1 r_2 \dots r_{m-2}}} p_{r_{m-1}}(\xi) d\xi \right. \\ &\quad \left. + \beta_{r_{m-1}} a_{r_1 r_2 \dots r_{m-2}} + \alpha_{r_{m-1}} \int_{I_{r_1 r_2 \dots r_{m-2}}} f_1(\xi) d\xi \right] = \dots \\ &= \sum_{k=1}^m \prod_{j=k+1}^m (|I_{r_j}| \alpha_{r_j}) |I_{r_k}| \left( \int_{I_{r_1 r_2 \dots r_{k-1}}} p_{r_k}(\xi) d\xi + \beta_{r_k} a_{r_1 r_2 \dots r_{k-1}} \right) + \prod_{j=1}^m (|I_{r_j}| \alpha_{r_j}) \int_0^1 f_1(\xi) d\xi. \end{aligned}$$

□

Since  $f_1(x)$  is continuous, the integral average  $b_{r_1 r_2 \dots r_m} / |I_{r_1 r_2 \dots r_m}|$  can be taken as a good approximation of  $f_1(x)$  in the subinterval  $I_{r_1 r_2 \dots r_m}$ , when  $m$  is very large, leading to the following definition of the approximating operator  $\mathcal{Q}_m$  on the interval  $I$ :

**Definition 3.1.** Let

$$\mathcal{Q}_m(f_1, x) = \sum_{r_1, r_2, \dots, r_m=1}^N \chi_{I_{r_1 r_2 \dots r_m}}(x) \frac{b_{r_1 r_2 \dots r_m}}{|I_{r_1 r_2 \dots r_m}|}, \quad (3.6)$$

where  $I_{r_1 r_2 \dots r_m}$  is defined by (3.4),  $b_{r_1 r_2 \dots r_m}$  is defined by (3.5), and

$$\chi_{I_{r_1 r_2 \dots r_m}}(x) = \begin{cases} 1 & x \in I_{r_1 r_2 \dots r_m}, \\ 0 & x \in I \setminus I_{r_1 r_2 \dots r_m}. \end{cases}$$

**Lemma 3.2.** As  $m \rightarrow \infty$ , the operator  $\mathcal{Q}_m(f_1, x)$ , given by (3.6), converges to  $f_1(x)$  uniformly on  $I$ .

*Proof.* The proof follows immediately by using Mean Value Theorem. □

The following notations are used in smoothness analysis:  $\alpha = \max\{|\alpha_i| : i = 1, 2, \dots, N\}$ ,  $\beta = \max\{|\beta_i| : i = 1, 2, \dots, N\}$ ,  $\gamma = \max\{|\gamma_i| : i = 1, 2, \dots, N\}$ ,  $\lambda = \min\{\lambda_i : i = 1, 2, \dots, N\}$ ,  $\Omega_i = \frac{|\alpha_i|}{|I_i|^\lambda}$ ,  $\Omega = \max\{\Omega_i : i = 1, 2, \dots, N\}$ ,  $\mu = \min\{\mu_i : i = 1, 2, \dots, N\}$ ,  $\Gamma_i = \frac{|\gamma_i|}{|I_i|^\mu}$ ,  $\Gamma = \max\{\Gamma_i : i = 1, 2, \dots, N\}$ ,  $\Theta_i = \frac{|\alpha_i|}{|I_i|^\mu}$ ,  $\Theta = \max\{\Theta_i : i = 1, 2, \dots, N\}$ ,  $|I_{\min}| = \min\{|I_i| : i = 1, 2, \dots, N\}$ ,  $|I_{\max}| = \max\{|I_i| : i = 1, 2, \dots, N\}$ , Modulus of continuity of  $f_1(x)$  as  $\omega(f_1, t) = \sup_{|h| \leq t} \sup_x |f_1(x+h) - f_1(x)|$ ,

and  $Lip \delta = \{f : |f(x) - f(y)| \leq M|x - y|^\delta, M > 0\}$ .

Using the above lemmas and notations, we now prove our smoothness results according to the magnitude of  $\Theta$ .

**Theorem 3.1.** Let  $f_1(x)$  be the CHFIF defined by (3.2) with  $\Theta < 1$ . Then, (a) for  $\Omega \neq 1$  and  $\Gamma \neq 1$ ,  $f_1 \in Lip \delta$  (b) for  $\Omega = 1$  or  $\Gamma = 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\delta \log |t|)$ , for suitable values of  $\delta \in (0, 1]$ .

*Proof.* In order to calculate the Hölder exponent of CHFIF  $f_1$ , a suitable upper bound on the difference between  $f_1(x)$  and  $f_1(\bar{x})$  for  $x, \bar{x} \in [0, 1]$  is needed to be found. In view of Lemma 3.2, it is sufficient to find an upper bound on the difference between functional values of their operator approximations  $\mathcal{Q}_m(f_1, x)$  and  $\mathcal{Q}_m(f_1, \bar{x})$ .

For  $0 \leq x < \bar{x} \leq 1$ , there exists a least  $m$  such that  $I_{r_1 r_2 \dots r_m}$  is the largest interval contained in  $[x, \bar{x}]$ . So, either  $x$  or  $\bar{x} \in I_{r_2 r_3 \dots r_m}$ . Assume that,  $x \in I_{sr_2 \dots r_m}$ ,  $s \leq r_1 - 1$ ,  $\bar{x} \in I_{tr_2 \dots r_m}$ ,  $t \geq r_1 + 1$ , or  $\bar{x} \in I_{t'r_2+1 \dots r_m}$ ,  $1 \leq t' \leq N$ . Let  $n, m \in \mathbb{N}$  and  $n > m$ . Taking further refinement of the above two intervals, we assume that  $x \in I_{u_1 \dots u_{n-m} sr_2 \dots r_m}$ ,  $\bar{x} \in I_{v_1 \dots v_{n-m} tr'_2 r_3 \dots r_m}$ . It now follows that

$$\begin{aligned} \mathcal{Q}_n(f_1, x) &= \frac{1}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_m}|} \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_m}} f_1(\xi) d\xi \\ &= \frac{1}{|I_{u_1}| |I_{u_2}| \dots |I_{u_{n-m}}| |I_s| |I_{r_2}| \dots |I_{r_m}|} \left[ \sum_{k=3}^m \prod_{j=k+1}^m (|I_{r_j}| \alpha_{r_j}) |I_{r_k}| \right. \\ &\quad \left. \left( \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}} p_{r_k}(\xi) d\xi + \beta_{r_k} a_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}} \right) + \prod_{j=3}^m (|I_{r_j}| \alpha_{r_j}) \int_{I_{u_1 \dots u_{n-m} sr_2}} f_1(\xi) d\xi \right] \\ &= \sum_{k=3}^m \left( \prod_{j=k+1}^m \alpha_{r_j} \right) \frac{1}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} \left( \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}} p_{r_k}(\xi) d\xi + \beta_{r_k} a_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}} \right) \\ &\quad + \left( \prod_{j=3}^m \alpha_{r_j} \right) \frac{1}{|I_{u_1 \dots u_{n-m} sr_2}|} \int_{I_{u_1 \dots u_{n-m} sr_2}} f_1(\xi) d\xi. \end{aligned}$$

Similarly, the expression for  $\mathcal{Q}_n(f_1, \bar{x})$  can be written as

$$\begin{aligned} \mathcal{Q}_n(f_1, \bar{x}) &= \sum_{k=3}^m \left( \prod_{j=k+1}^m \alpha_{r_j} \right) \frac{1}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}|} \left( \int_{I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}} p_{r_k}(\xi) d\xi + \beta_{r_k} a_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}} \right) \\ &\quad + \left( \prod_{j=3}^m \alpha_{r_j} \right) \frac{1}{|I_{v_1 \dots v_{n-m} tr'_2}|} \int_{I_{v_1 \dots v_{n-m} tr'_2}} f_1(\xi) d\xi. \end{aligned}$$

To estimate  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})|$ , observe that

$$\begin{aligned} &\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x}) \\ &= \sum_{k=3}^m \left( \prod_{j=k+1}^m \alpha_{r_j} \right) \left[ \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}} \frac{p_{r_k}(\xi) d\xi}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} - \int_{I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}} \frac{p_{r_k}(\xi) d\xi}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}|} \right] \\ &\quad + \left( \prod_{j=3}^m \alpha_{r_j} \right) \left[ \int_{I_{u_1 \dots u_{n-m} sr_2}} \frac{f_1(\xi) d\xi}{|I_{u_1 \dots u_{n-m} sr_2}|} - \int_{I_{v_1 \dots v_{n-m} tr'_2}} \frac{f_1(\xi) d\xi}{|I_{v_1 \dots v_{n-m} tr'_2}|} \right] \\ &\quad + \sum_{k=3}^m \left( \prod_{j=k+1}^m \alpha_{r_j} \right) \beta_{r_k} \left[ \frac{a_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} - \frac{a_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}|} \right]. \end{aligned} \tag{3.7}$$

From the approximation by Gang [20],

$$\begin{aligned} a_{r_1 r_2 \dots r_m} &= \int_{I_{r_1 r_2 \dots r_m}} f_2(\xi) d\xi \\ &= \sum_{k=1}^m \prod_{j=k+1}^m (|I_{r_j}| \gamma_{r_j}) |I_{r_k}| \int_{I_{r_1 r_2 \dots r_{k-1}}} q_{r_k}(\xi) d\xi + \prod_{j=1}^m (|I_{r_j}| \gamma_{r_j}) \int_0^1 f_2(\xi) d\xi, \end{aligned}$$

it follows that

$$\begin{aligned} \frac{a_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} &= \frac{1}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}} f_2(\xi) d\xi \\ &= \frac{1}{|I_{u_1 \dots u_{n-m} sr_2}||I_{r_3}| \dots |I_{r_{k-1}}|} \left[ \sum_{l=3}^{k-1} \prod_{i=l+1}^{k-1} (|I_{r_i}| \gamma_{r_i}) |I_{r_l}| \right. \\ &\quad \cdot \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}} q_{r_l}(\xi) d\xi + \prod_{i=3}^{k-1} (|I_{r_i}| \gamma_{r_i}) \int_{I_{u_1 \dots u_{n-m} sr_2}} f_2(\xi) d\xi \left. \right] \\ &= \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} \gamma_{r_i} \right) \frac{1}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}|} \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}} q_{r_l}(\xi) d\xi \\ &\quad + \left( \prod_{i=3}^{k-1} \gamma_{r_i} \right) \frac{1}{|I_{u_1 \dots u_{n-m} sr_2}|} \int_{I_{u_1 \dots u_{n-m} sr_2}} f_2(\xi) d\xi. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{a_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}|} &= \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} \gamma_{r_i} \right) \frac{1}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{l-1}}|} \int_{I_{v_1 \dots v_{n-m} tr'_2 \dots r_{l-1}}} q_{r_l}(\xi) d\xi \\ &\quad + \left( \prod_{i=3}^{k-1} \gamma_{r_i} \right) \frac{1}{|I_{v_1 \dots v_{n-m} tr'_2}|} \int_{I_{v_1 \dots v_{n-m} tr'_2}} f_2(\xi) d\xi. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left| \frac{a_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} - \frac{a_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}|} \right| \\ &= \left| \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} \gamma_{r_i} \right) \left[ \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}} \frac{q_{r_l}(\xi) d\xi}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}|} - \int_{I_{v_1 \dots v_{n-m} tr'_2 \dots r_{l-1}}} \frac{q_{r_l}(\xi) d\xi}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{l-1}}|} \right] \right. \\ &\quad \left. + \left( \prod_{i=3}^{k-1} \gamma_{r_i} \right) \left[ \int_{I_{u_1 \dots u_{n-m} sr_2}} \frac{f_2(\xi) d\xi}{|I_{u_1 \dots u_{n-m} sr_2}|} - \int_{I_{v_1 \dots v_{n-m} tr'_2}} \frac{f_2(\xi) d\xi}{|I_{v_1 \dots v_{n-m} tr'_2}|} \right] \right| \\ &\leq \sum_{l=3}^{k-1} \prod_{i=l+1}^{k-1} |\gamma_{r_i}| \left| \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}} \frac{q_{r_l}(\xi) - q_{r_l}(x_{r_3 \dots r_{l-1}})}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{l-1}}|} d\xi \right| \\ &\quad + \left| \int_{I_{v_1 \dots v_{n-m} tr'_2 \dots r_{l-1}}} \frac{q_{r_l}(\xi) - q_{r_l}(x_{r_3 \dots r_{l-1}})}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{l-1}}|} d\xi \right| + \left( \prod_{i=3}^{k-1} \gamma_{r_i} \right) \cdot 2 \|f_2\|_{\infty} \\ &\leq \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} |\gamma_{r_i}| \right) M_1 |I_{r_3 \dots r_{l-1}}|^{\mu_{r_l}} + M_2 \left( \prod_{i=3}^{k-1} \gamma_{r_i} \right). \end{aligned} \tag{3.8}$$

where  $M_1$  is Lipschitz bound and  $M_2 = 2\|f_2\|_{\infty}$ . Using (3.8) in (3.7),

$$\begin{aligned} |\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| &\leq \sum_{k=3}^m \left( \prod_{j=k+1}^m |\alpha_{r_j}| \right) \left| \int_{I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}} \frac{p_{r_k}(\xi) - p_{r_k}(x_{r_3 \dots r_{k-1}})}{|I_{u_1 \dots u_{n-m} sr_2 \dots r_{k-1}}|} d\xi \right| \\ &\quad + \left| \int_{I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}} \frac{p_{r_k}(\xi) - p_{r_k}(x_{r_3 \dots r_{k-1}})}{|I_{v_1 \dots v_{n-m} tr'_2 \dots r_{k-1}}|} d\xi \right| + \left( \prod_{j=3}^m |\alpha_{r_j}| \right) \cdot 2 \|f_1\|_{\infty} \\ &\quad + \sum_{k=3}^m \left( \prod_{j=k+1}^m |\alpha_{r_j}| \right) |\beta_{r_k}| \left[ \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} |\gamma_{r_i}| \right) M_1 |I_{r_3 \dots r_{l-1}}|^{\mu_{r_l}} + M_2 \left( \prod_{i=3}^{k-1} |\gamma_{r_i}| \right) \right] \end{aligned}$$

$$\begin{aligned} &\leq \sum_{k=3}^m \left( \prod_{j=k+1}^m |\alpha_{r_j}| \right) M_3 |I_{r_3 \dots r_{k-1}}|^{\lambda_{r_k}} + M_4 \left( \prod_{j=3}^m |\alpha_{r_j}| \right) + \sum_{k=3}^m \left( \prod_{j=k+1}^m |\alpha_{r_j}| \right) |\beta_{r_k}| \cdot \\ &\quad \left[ \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} |\gamma_{r_i}| \right) M_1 |I_{r_3 \dots r_{l-1}}|^{\mu_{r_l}} + M_2 \left( \prod_{i=3}^{k-1} |\gamma_{r_i}| \right) \right], \end{aligned}$$

where  $M_3$  is Lipschitz bound, and  $M_4 = 2\|f_1\|_\infty$ . From the above inequality it follows that

$$\begin{aligned} |\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| &\leq M_3 \sum_{k=3}^m \left( \prod_{j=k+1}^m |\alpha_{r_j}| \right) \frac{\prod_{i'=3}^m |I_{r_{i'}}|^{\lambda_{r_k}}}{\prod_{j'=k}^m |I_{r_{j'}}|^{\lambda_{r_k}}} + M_4 \left( \prod_{j=3}^m |\alpha_{r_j}| \right) \\ &+ \sum_{k=3}^m \left( \prod_{j=k+1}^m |\alpha_{r_j}| \right) |\beta_{r_k}| \cdot \left[ M_1 \sum_{l=3}^{k-1} \left( \prod_{i=l+1}^{k-1} |\gamma_{r_i}| \right) \frac{\prod_{i'=3}^m |I_{r_{i'}}|^{\mu_{r_l}}}{\prod_{j'=l}^m |I_{r_{j'}}|^{\mu_{r_l}}} + M_2 \left( \prod_{i=3}^{k-1} |\gamma_{r_i}| \right) \right] \\ &\leq \frac{M_3}{(|I_{\min}|^\lambda)^3} \left( \prod_{i'=1}^m |I_{r_{i'}}|^\lambda \right) \sum_{k=3}^m \prod_{j=k+1}^m \frac{|\alpha_{r_j}|}{|I_{r_j}|^\lambda} + \frac{M_4}{(|I_{\min}|^\lambda)^2} \left( \prod_{j=1}^m |I_{r_j}|^\lambda \right) \prod_{j=3}^m \frac{|\alpha_{r_j}|}{|I_{r_j}|^\lambda} + \sum_{k=3}^m |\beta_{r_k}| \prod_{j=k+1}^m |\alpha_{r_j}| \cdot \\ &\quad \left[ \frac{M_1}{(|I_{\min}|^\lambda)^3} \left( \prod_{i'=1}^m |I_{r_{i'}}|^\mu \right) \sum_{l=3}^{k-1} \prod_{i=l+1}^{k-1} \frac{|\gamma_{r_i}|}{|I_{r_i}|^\mu} + \frac{M_2}{(|I_{\min}|^\lambda)^2} \left( \prod_{i=1}^m |I_{r_i}|^\mu \right) \prod_{i=3}^{k-1} \frac{|\gamma_{r_i}|}{|I_{r_i}|^\mu} \right] \cdot \prod_{j=k}^m \frac{1}{|I_{r_j}|^\mu} \\ &\leq M_5 \left( \prod_{i=1}^m |I_{r_i}|^\lambda \right) \left[ \sum_{k=3}^m \prod_{j=k+1}^m \Omega_{r_j} + \prod_{j=3}^m \Omega_{r_j} \right] + M_6 \sum_{k=3}^m |\beta_{r_k}| \prod_{j=k+1}^m \frac{|\alpha_{r_j}|}{|I_{r_j}|^\mu} \cdot \\ &\quad \left( \prod_{i=1}^m |I_{r_i}|^\mu \right) \left[ \sum_{l=3}^{k-1} \prod_{i=l+1}^{k-1} \Gamma_{r_i} + \prod_{i=3}^{k-1} \Gamma_{r_i} \right], \end{aligned}$$

where  $M_5 = \text{Max}\left\{ \frac{M_3}{(|I_{\min}|^\lambda)^3}, \frac{M_4}{(|I_{\min}|^\lambda)^2} \right\}$  and  $M_6 = \text{Max}\left\{ \frac{M_1}{(|I_{\min}|^\lambda)^4}, \frac{M_2}{(|I_{\min}|^\lambda)^3} \right\}$ . The above inequality gives

$$|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^\lambda \left( \sum_{k=2}^m \Omega^{m-k} \right) + M_6 \beta |x - \bar{x}|^\mu \sum_{k=3}^m \Theta^{m-k} \left( \sum_{l=3}^{k-1} \Gamma^{k-l} \right). \quad (3.9)$$

Since  $\Theta < 1$ , (3.9) further reduces to

$$|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^\lambda \left( \sum_{k=2}^m \Omega^{m-k} \right) + M_6 \frac{\beta}{1 - \Theta} |x - \bar{x}|^\mu \left( \sum_{k=1}^{m-3} \Gamma^k \right). \quad (3.10)$$

**Case (a).**  $\Omega \neq 1$  and  $\Gamma \neq 1$ : The desired Hölder exponents are found individually for each of the following subcases:

**I.**  $\Omega < 1$  and  $\Gamma < 1$ :  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq \frac{M_5}{1-\Omega} |x - \bar{x}|^\lambda + \frac{M_6 \beta}{(1-\Theta)(1-\Gamma)} |x - \bar{x}|^\mu \leq M_7 |x - \bar{x}|^{\delta_1}$ , where  $M_7 = \max\left\{ \frac{M_5}{1-\Omega}, \frac{M_6 \beta}{(1-\Theta)(1-\Gamma)} \right\}$  and  $\delta_1 = \min(\lambda, \mu)$ . Thus, as  $n \rightarrow \infty$ , the above inequality together with Lemma 3.2 gives  $f_1 \in \text{Lip}\delta$  with  $\delta = \delta_1$ .

**II.**  $\Omega > 1$  and  $\Gamma > 1$ :

$$|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^\lambda m \Omega^m + \frac{M_6 \beta}{1 - \Theta} |x - \bar{x}|^\mu m \Gamma^m. \quad (3.11)$$

Suppose  $\tau_1 > 0$  such that  $|x - \bar{x}|^\lambda m \Omega^m \leq |x - \bar{x}|^{\tau_1}$ . Then,

$$\tau_1 \leq \lambda + \frac{m \log \Omega}{\log |x - \bar{x}|}. \quad (3.12)$$

Further,  $|I_{r_1 \dots r_m}| \leq |x - \bar{x}| < 1 \Rightarrow |I_{\min}|^m \leq |x - \bar{x}| \Rightarrow \frac{1}{m \log |I_{\min}|} \geq \frac{1}{\log |x - \bar{x}|}$ .

Also,  $\Omega \leq \frac{\alpha}{|I_{\min}|^\lambda} \Rightarrow \log \Omega \leq \log \alpha - \lambda \log |I_{\min}|$ . Therefore, by (3.12),  $\tau_1 \leq \frac{\log \alpha}{\log |I_{\min}|}$ .

Similarly, if  $\tau_2 > 0$  is such that  $|x - \bar{x}|^\mu m \Gamma^m \leq |x - \bar{x}|^{\tau_2}$ , then  $\tau_2 \leq \frac{\log \gamma}{\log |I_{\min}|}$ .

Let  $\tau_3 = \min\{\frac{\log \alpha}{\log |I_{\min}|}, \frac{\log \gamma}{\log |I_{\min}|}\}$ . From (3.11), for any  $\delta_2 \leq \tau_3$ ,  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_8 |x - \bar{x}|^{\delta_2}$ , where  $M_8 = \max\{M_5, \frac{M_6 \beta}{1-\Theta}\}$ . Now, the last inequality together with Lemma 3.2 gives  $f_1 \in Lip\delta$  with  $\delta = \delta_2$ .

**III.**  $\Omega > 1$  and  $\Gamma < 1$  :  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^{\tau_1} + \frac{M_6 \beta}{(1-\Theta)(1-\Gamma)} |x - \bar{x}|^\mu \leq M_9 |x - \bar{x}|^{\delta_3}$ , where  $M_9 = \max\{M_5, \frac{M_6 \beta}{(1-\Theta)(1-\Gamma)}\}$  and  $\delta_3 = \min(\tau_1, \mu)$ . Thus, as  $n \rightarrow \infty$ , the last inequality together with Lemma 3.2 gives  $f_1 \in Lip\delta$  with  $\delta = \delta_3$ .

**IV.**  $\Omega < 1$  and  $\Gamma > 1$  :  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq \frac{M_5}{1-\Omega} |x - \bar{x}|^\lambda + \frac{M_6 \beta}{1-\Theta} |x - \bar{x}|^{\tau_2} \leq M_{10} |x - \bar{x}|^{\delta_4}$ , where  $M_{10} = \max\{\frac{M_5}{1-\Omega}, \frac{M_6 \beta}{1-\Theta}\}$  and  $\delta_4 = \min(\lambda, \tau_2)$ . So, as  $n \rightarrow \infty$ , the above inequality together with Lemma 3.2 gives  $f_1 \in Lip\delta$  with  $\delta = \delta_4$ .

**Case (b).**  $\Omega = 1$  or  $\Gamma = 1$ : The desired Hölder exponents are found individually for each of the following subcases:

**I.**  $\Omega = 1$  and  $\Gamma \leq 1$  or  $\Omega < 1$  and  $\Gamma = 1$  : For  $\Omega = 1$  and  $\Gamma = 1$ ,

$$\begin{aligned} |\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| &\leq (M_5 |x - \bar{x}|^\lambda + \frac{M_6 \beta}{1 - \Theta} |x - \bar{x}|^\mu) \cdot (m - 1) \\ &\leq (M_5 |x - \bar{x}|^\lambda + \frac{M_6 \beta}{1 - \Theta} |x - \bar{x}|^\mu) \frac{\log |x - \bar{x}|}{\log |I_{\max}|} \\ &\leq M_{11} (\log |x - \bar{x}|) |x - \bar{x}|^{\delta_1}, \end{aligned}$$

where  $M_{11} = \frac{M_8}{\log |I_{\max}|}$ . As  $n \rightarrow \infty$ , the last inequality together with Lemma 3.2 gives  $\omega(f_1, t) = \mathcal{O}(|t|^\delta \log |t|)$  with  $\delta = \delta_1$ . For  $\Omega = 1$  and  $\Gamma < 1$ ,  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq \frac{M_5}{\log |I_{\max}|} |x - \bar{x}|^\lambda \log |x - \bar{x}| + \frac{M_6 \beta}{(1-\Theta)(1-\Gamma)} |x - \bar{x}|^\mu \leq M_{12} |x - \bar{x}|^{\delta_1} (1 + \log |x - \bar{x}|)$ , where  $M_{12} = \max\{\frac{M_5}{\log |I_{\max}|}, \frac{M_6 \beta}{(1-\Theta)(1-\Gamma)}\}$ . Hence, as  $n \rightarrow \infty$ , the above inequality together with Lemma 3.2 gives  $\omega(f_1, t) = \mathcal{O}(|t|^\delta (1 + \log |t|)) \equiv \mathcal{O}(|t|^\delta \log |t|)$  with  $\delta = \delta_1$ . The estimate for  $\Omega < 1$  and  $\Gamma = 1$  follows using analogous arguments.

**II.**  $\Omega > 1$  and  $\Gamma = 1$  :  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^{\tau_1} + \frac{M_6 \beta}{(1-\Theta) \log |I_{\max}|} |x - \bar{x}|^\mu \log |x - \bar{x}| \leq M_{13} |x - \bar{x}|^{\delta_3} (1 + \log |x - \bar{x}|)$ , where  $M_{13} = \max\{M_5, \frac{M_6 \beta}{(1-\Theta) \log |I_{\max}|}\}$ . Making  $n \rightarrow \infty$ , the above inequality together with Lemma 3.2 gives  $\omega(f_1, t) = \mathcal{O}(|t|^\delta \log |t|)$  with  $\delta = \delta_3$ .

**III.**  $\Omega = 1$  and  $\Gamma > 1$  :  $|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq \frac{M_5}{\log |I_{\max}|} |x - \bar{x}|^\lambda \log |x - \bar{x}| + \frac{M_6 \beta}{1-\Theta} |x - \bar{x}|^{\tau_2} \leq M_{14} |x - \bar{x}|^{\delta_4} (1 + \log |x - \bar{x}|)$ , where  $M_{14} = \max\{\frac{M_5}{\log |I_{\max}|}, \frac{M_6 \beta}{1-\Theta}\}$ . So, as  $n \rightarrow \infty$ , the above inequality together with Lemma 3.2 gives  $\omega(f_1, t) = \mathcal{O}(|t|^\delta \log |t|)$  with  $\delta = \delta_4$ .

Theorem 3.1 now follows from the above cases with suitable  $\delta$  as found in various subcases. □

The smoothness results for the class of CHFIFs when  $\Theta = 1$  are given by the following:

**Theorem 3.2.** Let  $f_1(x)$  be the CHFIF defined by (3.2) with  $\Theta = 1$ . Then, (a) for  $\Omega \neq 1$  and  $\Gamma \neq 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\delta \log |t|)$  (b) for  $\Omega = 1$  or  $\Gamma = 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\delta (\log |t|)^2)$ , for suitable values of  $\delta \in (0, 1]$ .

*Proof.* Since  $\Theta = 1$ , (3.9) gives,

$$|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^\lambda \left(\sum_{k=2}^m \Omega^{m-k}\right) + \frac{M_6 \beta}{\log |I_{\max}|} |x - \bar{x}|^\mu \log |x - \bar{x}| \left(\sum_{k=1}^{m-3} \Gamma^k\right). \quad (3.13)$$

The rest of proof is similar to that of Theorem 3.1 with the respective values of  $\delta$  as in different cases of Theorem 3.1. □

Finally, the smoothness results for the class of CHFIFs for  $\Theta > 1$  are given by the following:

**Theorem 3.3.** Let  $f_1(x)$  be the CHFIF defined by (3.2) with  $\Theta > 1$ . Then, (a) for  $\Omega \neq 1$  and  $\Gamma \neq 1$ ,  $f_1 \in Lip\delta$  (b) for  $\Omega = 1$  or  $\Gamma = 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\delta \log |t|)$ , for suitable values of  $\delta \in (0, 1]$ .

*Proof.* Inequality (3.9) for  $\Theta > 1$  gives

$$|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^\lambda \left( \sum_{k=2}^m \Omega^{m-k} \right) + M_6 \beta |x - \bar{x}|^\mu \cdot m \Theta^m \cdot \left( \sum_{k=1}^{m-3} \Gamma^k \right). \quad (3.14)$$

Let  $\tau_4 > 0$  be such that  $|x - \bar{x}|^\mu m \Theta^m \leq |x - \bar{x}|^{\tau_4}$ . Then,

$$\tau_4 \leq \mu + \frac{m \log \Theta}{\log |x - \bar{x}|} \leq \frac{\log \alpha}{\log |I_{\min}|}.$$

Since  $\tau_1$  in Theorem 3.1 satisfies  $\tau_1 \leq \frac{\log \alpha}{\log |I_{\min}|}$ , we can choose  $\tau_4 = \tau_1$  so that (3.14) reduces to

$$|\mathcal{Q}_n(f_1, x) - \mathcal{Q}_n(f_1, \bar{x})| \leq M_5 |x - \bar{x}|^\lambda \left( \sum_{k=2}^m \Omega^{m-k} \right) + M_6 \beta |x - \bar{x}|^{\tau_1} \left( \sum_{k=1}^{m-3} \Gamma^k \right). \quad (3.15)$$

The rest of the proof is similar to that of Theorem 3.1, by considering (3.15) in place of (3.10). As in Theorem 3.1, the value of  $\delta$  in different cases are given by Case (a): I.  $\delta = \delta_5 = \min(\lambda, \tau_1)$ , II.  $\delta = \delta_6$  where  $\delta_6 \leq \frac{\log \alpha \gamma}{\log |I_{\min}|} - \mu$ , III.  $\delta = \delta_7$  where  $\delta_7 \leq \frac{\log \alpha}{\log |I_{\min}|}$ , IV.  $\delta = \delta_8 = \min(\lambda, \delta_6)$ , and Case (b): I.  $\delta = \delta_5$ , II.  $\delta = \delta_7$ , III.  $\delta = \delta_8$ .  $\square$

**Remark 3.1.** 1. It follows from Theorems 3.1-3.3, that the smoothness of the CHFIF depends on the free variables  $\alpha_i, \gamma_i$  and the Lipschitz exponents  $\lambda_i$  and  $\mu_i$  but independent of the constrained free variables  $\beta_i$ .  
 2. Let  $\lambda = \mu$  and  $\Theta < 1$ . Then,  $\Omega < 1$ . Theorem 3.1 now gives the following smoothness results depending on the magnitude of  $\Gamma$  for CHFIF  $f_1(x)$ . (A) For  $\Gamma < 1$ ,  $f_1(x) \in Lip\mu$ , since  $\delta_1 = \mu$  in this case. (B) For  $\Gamma = 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\mu \log |t|)$ , since  $\delta_1 = \mu$  in this case. (C) For  $\Gamma > 1$ ,  $f_1 \in Lip\tau_2$ , where  $\tau_2 \leq \frac{\log \gamma}{\log |I_{\min}|}$ , since  $\delta_4 = \min(\lambda, \tau_2) \leq \tau_2$  in this case.  
 3. Suppose  $\lambda = \mu$  and  $\Theta = 1$ . Then,  $\Omega = 1$ . Theorem 3.2 in this case gives the smoothness result as follows: (A) For  $\Gamma < 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\mu (\log |t|)^2)$ , since  $\delta_1 = \mu$  in this case. (B) For  $\Gamma = 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^\mu (\log |t|)^2)$ , since  $\delta_1 = \mu$  in this case. (C) For  $\Gamma > 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^{\delta_4} (\log |t|)^2)$ , where  $\delta_4 = \min(\lambda, \tau_2) \leq \tau_2$ .  
 4. Let  $\lambda = \mu$  and  $\Theta > 1$ . Then,  $\Theta = \frac{\max\{\alpha_i : i=1,2,\dots,N\}}{|I_{\min}|^\mu} > 1$  which in turn implies  $\frac{\log \alpha}{\log |I_{\min}|} < \mu$ . Since  $\delta_6 \leq \frac{\log \gamma}{\log |I_{\min}|} + \left( \frac{\log \alpha}{\log |I_{\min}|} - \mu \right) < \frac{\log \gamma}{\log |I_{\min}|}$  and  $\tau_2 \leq \frac{\log \gamma}{\log |I_{\min}|}$ , we may choose  $\delta_6 \leq \tau_2$ . Further,  $\tau_1 \leq \frac{\log \alpha}{\log |I_{\min}|}$  implies  $\tau_1 < \mu$ . With these inequalities, the smoothness results as derived from Theorem 3.3 in the case  $\lambda = \mu$  and  $\Theta = \Omega > 1$  are as follows: (A) For  $\Gamma < 1$ ,  $f_1 \in Lip\tau_1 \supseteq Lip\mu$ . (B) For  $\Gamma = 1$ ,  $\omega(f_1, t) = \mathcal{O}(|t|^{\tau_1} \log |t|)$  which gives  $\omega(f_1, t) = \mathcal{O}(|t|^\mu \log |t|)$ . (C) For  $\Gamma > 1$ ,  $f_1 \in Lip\delta_6 \supseteq Lip\tau_2$ .  
 5. If  $f_1(x) = f_2(x)$ , then  $f_1(x)$  is also self-affine and in such case,  $y_i = z_i$ ,  $\alpha_i + \beta_i = \gamma_i$ , and  $p_i(x) = q_i(x)$ . Hence,  $\lambda_i = \mu_i \Rightarrow \lambda = \mu \Rightarrow \delta_1 = \mu$  and  $\Omega = \Theta$ . For self-affine function  $f_1(x) = f_2(x)$ ,  $f_1$  belongs to the intersection of the function spaces occurring for the same case((A), (B), or (C)) of Remarks 3-5 as above. We note that the intersection of these function spaces is independent of  $\Theta$ . Since, the class of CHFIFs for  $\Theta < 1$  is contained in the class of CHFIFs for  $\Theta = 1$  and  $\Theta > 1$ , the smoothness results in [20] for self-affine function  $f_2(x)$  follows as special case of our smoothness results derived in the above Remarks 2-4.

## 4 Fractal dimension of CHFIF

The following definitions are needed in the sequel: The conditions  $\Omega = 1$ ,  $\Gamma = 1$ , or  $\Theta = 1$  are called *critical conditions*. The CHFIF  $f_1(x)$  with any one of these condition is called *critical CHFIF*. Let  $\mathcal{N}(A, \epsilon)$  be the smallest number of closed balls of radius  $\epsilon > 0$ , needed to cover A. Then, the Fractal dimension of A is defined by  $D_B(A) = \lim_{\epsilon \rightarrow 0} \frac{\log \mathcal{N}(A, \epsilon)}{-\log \epsilon}$ , whenever the limit exists. Our following theorems give bounds on the fractal dimension for the critical CHFIFs where the range or oscillation of  $f_1$  is taken over an interval.

**Theorem 4.1.** Let fractal dimension of CHFIF  $f_1(x)$ , defined by (3.2) exists. Then, for the critical condition  $\Omega = 1$ ,

$$1 - \frac{\log \sum_{k=1}^N |\alpha_k|}{\log |I_{\max}|} \leq D_B(\text{graph}(f_1)) \leq 1 - \delta - \frac{\log N}{\log |I_{\max}|}, \quad (4.1)$$

and for the critical condition  $\Gamma = 1$ ,

$$1 - \frac{\log \sum_{k=1}^N |\gamma_k|}{\log |I_{\max}|} \leq D_B(\text{graph}(f_1)) \leq 1 - \delta - \frac{\log N}{\log |I_{\max}|}, \quad (4.2)$$

where  $\delta$  takes suitable values as in the subcases in Theorems 3.1-3.3.

*Proof.* Let  $\Theta < 1$  and  $\Omega = 1$ . Since  $\omega(f_1, t) = \mathcal{O}(|t|^{\delta_1} \log |t|)$ , (cf. Theorem 3.1), for all  $x \neq x^*$ ,  $x, x^* \in I$ , there exist constants  $C_1, C_2$  such that

$$C_1 |x - x^*|^{\delta_1} \leq |f_1(x) - f_1(x^*)| \leq C_2 |x - x^*|^{\delta_1} \log |x - x^*|. \quad (4.3)$$

Suppose,  $G_{r_1, r_2, \dots, r_m} = \{(x, f_1(x), f_2(x)) \mid x \in I_{r_1, r_2, \dots, r_m}\}$ . Define,  $|A|_X = \sup\{|x - \bar{x}| \mid (x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in A\}$ ,  $|A|_Y = \sup\{|y - \bar{y}| \mid (x, y, z), (\bar{x}, \bar{y}, \bar{z}) \in A\}$ , for any  $A \subset \mathbb{R}^3$ . Since,  $|G_{r_1, r_2, \dots, r_m}|_X = |I_{r_1, r_2, \dots, r_m}|$ , (4.3) reduces to

$$C_1 |I_{r_1, r_2, \dots, r_m}|^{\delta_1} \leq |G_{r_1, r_2, \dots, r_m}|_Y \leq C_2 |I_{r_1, r_2, \dots, r_m}|^{\delta_1} \log |I_{r_1, r_2, \dots, r_m}|. \quad (4.4)$$

Choose  $m$  large such that  $|I_{\max}|^m < \frac{1}{2}\epsilon$ ,  $\epsilon > 0$ . Since,  $\Omega_{r_j} \leq \Omega = 1$  implies  $|\alpha_{r_j}| \leq |I_{r_j}|^\lambda \leq |I_{r_j}|^{\delta_1}$  and  $|I_{r_1, r_2, \dots, r_m}| = |I_{r_1}| \cdot |I_{r_2}| \dots |I_{r_m}|$ , it follows by (4.4) that

$$C_1 |\alpha_{r_1}| \cdot |\alpha_{r_2}| \dots |\alpha_{r_m}| \leq |G_{r_1, r_2, \dots, r_m}|_Y \leq C_2 |I_{\max}|^{m\delta_1} \cdot m \log |I_{\max}|. \quad (4.5)$$

Taking summation over  $r_1, r_2, \dots, r_m$  from 1 to  $N$  in (4.5),

$$\begin{aligned} \sum_{r_1, r_2, \dots, r_m} C_1 |\alpha_{r_1}| \cdot |\alpha_{r_2}| \dots |\alpha_{r_m}| |I_{\max}|^{-m} &\leq \sum_{r_1, r_2, \dots, r_m} |G_{r_1, r_2, \dots, r_m}|_Y |I_{\max}|^{-m} \\ &\leq \sum_{r_1, r_2, \dots, r_m} C_2 |I_{\max}|^{m(\delta_1-1)} \cdot m \log |I_{\max}|. \end{aligned}$$

The above inequalities can be rewritten as

$$\bar{C}_1 |I_{\max}|^{-m} (|\alpha_1| + \dots + |\alpha_N|)^m \leq \mathcal{N}(\text{graph}(f_1), \epsilon) \leq \bar{C}_2 |I_{\max}|^{m(\delta_1-1)} \cdot m \log |I_{\max}| \cdot N^m.$$

The inequalities (4.1) follow from the last inequalities with  $\delta = \delta_1$ . The proof of (4.2) for  $\Theta < 1$ ,  $\Gamma = 1$  is similar to the above case.

Let  $\Theta = 1$  and  $\Omega = 1$ . Since  $\omega(f_1, t) = \mathcal{O}(|t|^{\delta_1} (\log |t|)^2)$  (cf. Theorem 3.2), for all  $x \neq x^*$ ,  $x, x^* \in I$ , there exist constants  $C_3, C_4$  such that

$$C_3 |x - x^*|^{\delta_1} \leq |f_1(x) - f_1(x^*)| \leq C_4 |x - x^*|^{\delta_1} (\log |x - x^*|)^2. \quad (4.6)$$

Now, using (4.6) in place of (4.3) the above arguments give that there are constants  $\bar{C}_3$  and  $\bar{C}_4$  such that

$$\bar{C}_3 |I_{\max}|^{-m} (|\alpha_1| + \dots + |\alpha_N|)^m \leq \mathcal{N}(\text{graph}(f_1), \epsilon) \leq \bar{C}_4 |I_{\max}|^{m(\delta_1-1)} \cdot (m \log |I_{\max}|)^2 \cdot N^m.$$

The proof of (4.1) for  $\Theta = 1$  and  $\Omega = 1$  follows from the above inequalities with  $\delta = \delta_1$ .

The proof of (4.1)-(4.2) is analogous in other cases.  $\square$

**Theorem 4.2.** Let fractal dimension of CHFIF  $f_1(x)$ , defined by (3.2) exists with  $\Theta = 1$ . Then, for  $\Omega \neq 1$  or  $\Gamma \neq 1$ ,

$$1 - \frac{\log \sum_{k=1}^N |\alpha_k|}{\log |I_{\max}|} \leq D_B(\text{graph}(f_1)) \leq 1 - \delta - \frac{\log N}{\log |I_{\max}|},$$

where  $\delta$  takes suitable values as in Theorem 3.2.

*Proof.* The proof is similar to the case  $\Theta < 1$  of Theorem 4.1.  $\square$

Theorems 4.1-4.2 lead to the following bounds on fractal dimension of equally spaced critical CHFIFs.

**Corollary 4.1.** *Let fractal dimension of CHFIF  $f_1(x)$ , defined by (3.2) exists. Then, for  $\Theta = 1$  or  $\Omega = 1$ ,*

$$1 + \frac{\log \sum_{k=1}^N |\alpha_k|}{\log N} \leq D_B(\text{graph}(f_1)) \leq 2 - \delta. \quad (4.7)$$

Further, for  $\Gamma = 1$ ,

$$1 + \frac{\log \sum_{k=1}^N |\gamma_k|}{\log N} \leq D_B(\text{graph}(f_1)) \leq 2 - \delta, \quad (4.8)$$

where  $\delta$  takes suitable values as in Theorems 3.1-3.3.

**Corollary 4.2.** *Let the equidistant CHFIF  $f_1(x)$  be defined by (3.2). Then,  $D_B(\text{graph}(f_1)) = 1$  in the following cases:*

1.  $\Theta \leq 1$ , either  $\delta = \delta_1 = 1$  or  $\delta = \delta_3 = 1$  and either  $\sum_{k=1}^N |\alpha_k| \leq 1$  or  $\sum_{k=1}^N |\gamma_k| \leq 1$ .
2.  $\Theta \leq 1$ ,  $\delta = \delta_4 = 1$ , with  $\sum_{k=1}^N |\alpha_k| \leq 1$  and for  $\Theta = 1$ ,  $\delta = \delta_2 = \tau_3 = 1$  with  $\sum_{k=1}^N |\alpha_k| \leq 1$ .
3.  $\Theta > 1$ , either  $\delta = \delta_5 = 1$  or  $\delta = \delta_8 = 1$  with  $\sum_{k=1}^N |\alpha_k| \leq 1$  and  $\delta = \delta_4 = 1$  with  $\sum_{k=1}^N |\gamma_k| \leq 1$ .

**Remark 4.1.** 1. *In the critical case  $\Gamma = 1$ , the fractal dimension bounds of CHFIF  $f_1(x)$  found in (4.2) coincide with the fractal dimension bounds of FIF  $f_2(x)$  found in [20], if  $f_1(x)$  is also self-affine.*

2. *In Corollary 4.2, critical CHFIF  $f_1(x)$  is considered as fractal function, since  $f_1(x)$  satisfies  $\omega(f_1, x) = O(|x|^\delta \log |x|)$ . Consequently, fractal functions having  $D_B(\text{graph}(f_1)) = 1$  can be constructed by using this corollary.*

## 5 Conclusion

In the present paper, we prove that the smoothness of CHFIF  $f_1(x)$  depends on free variables  $\alpha_i$  and  $\gamma_i$  as well as on the smoothness of  $p_i(x)$  and  $q_i(x)$ . Although,  $z_i$  and  $\beta_i$  are responsible for the shape of the CHFIF, these are found not to affect its smoothness. In general, the deterministic construction of functions  $f_1$  having order of modulus of continuity  $\omega(f_1, t) = O(|t|^\delta (\log |t|)^m)$  is possible through the CHFIF, where  $m$  is a non-negative integer and  $0 < \delta \leq 1$ . The fact that CHFIFs are different in shape although they are in the same function spaces may enable considering them in more general function spaces such as Besov and Triebel-Lizorkin spaces apart from Lipschitz spaces. These former spaces have additional indices that 'fine-tune' a function. Finally, the bounds of fractal dimension of CHFIFs are found in different critical conditions.

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