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New Exact and Explicit Traveling Wave Solutions for the CD and PKP Equations

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Abstract: In this paper, we establish exact solutions for nonlinear evolution equations. The homogeneous balance method is used to construct exact travelling wave solutions of the Calogero-Degasperis (CD) and Potential Kadomstev-Petviashvili (PKP) equations, in which the homogeneous balance method is applied to solve the Riccati equation and the reduced nonlinear ordinary differential equation. Many new families of exact travelling wave solutions of the Calogero and PKP equations are successfully obtained. The solutions obtained may be of significant importance for the explanation of some practical physical problems.

Key words:travelling wave solutions; periodic solutions; soliton solutions; homogeneous balance method; Calogero-Degasperis equation; potential Kadomstev-Petviashvili equation

1 Introduction

The world around us is inherently nonlinear. Nonlinear evolution equations (NEEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics [1]. Particularly, various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear PDEs. One of the basic physical problems for those models is to obtain their traveling wave solutions. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. The concepts like solitons, peakons, kinks, breathers, cusps and compactons are now thoroughly investigated in [11, 12]. During the past decades, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed among which are a variety of powerful methods, such as inverse scattering method [1], bilinear transformation [6], the tanh-sech method [7, 14, 15, 16], extended tanh method [4, 20, 21], sine-cosine method [17, 18, 19], homogeneous balance method [3, 13], pseudo spectral method [8], Lie group analysis [10] and the trial function have been used to investigate nonlinear dispersive and dissipative problems.

The CD equation or breaking soliton equation

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy} = 0, \tag{1}$$

is an important nonlinear wave equation. Eq. (1) was first established by Calogero and Degasperis [2] and is used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. The Hamiltonian structure, the Lax pair with non-isospectral problem and the Painleve property have been discussed. Using symbolic computation, new families of soliton-like solutions are obtained for (2+1)-dimensional breaking soliton equations using an ansatz [24]. With the help

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of symbolic computation, sixteen kinds of new special exact soliton-like solutions of (2+1)-dimensional breaking soliton equation are obtained by further generalized projective Riccati equation method [25].

Solitary waves are wave packets or pulses which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. A soliton is a very special type of solitary wave, which also keeps its waveform after collision with other solitons.

We next consider the PKP equation

$$u_{xt} + \frac{3}{2}u_x u_{xx} + \frac{1}{4}u_{xxxx} + \frac{3}{4}u_{yy} = 0. \tag{2}$$

New soliton-like solutions are obtained for (2+1)-dimensional PKP equation by using the symbolic computation method developed by Gao and Tian [5]. Solitary wave solutions obtained are merely a special case of the paper [9]. The nonlinear KP(n,n) equations and its variants are analytically studied by using the tanh method [22] and by using the sine-cosine method [23].

2 Exact and explicit traveling wave solutions for the CD equation

We can obtain more solutions of the ansatz equation-the Riccati equation, and therefore some new and more general types of solutions of CD Eq. (1) can be successfully given. The detailed procedure is as follows:

$$u = u(\xi), \quad \xi = x + y - \beta t. \tag{3}$$

Substituting Eq. (3) into Eq. (1) reduces Eq. (1) to the following nonlinear ordinary differential equation,

$$-\beta U'' - 6U'U'' + U^{(4)} = 0. \tag{4}$$

Integrating Eq. (4) once and chosing the constants of integration to be zero, we find

$$-\beta U' - 3(U')^2 + U''' = 0. \tag{5}$$

Let as suppose that Eq. (5) has solutions in the form

$$U = \sum_{i=0}^{m} q_i \phi^i, \tag{6}$$

where ϕ satisfies the Riccati equation

$$\phi' = a\phi^2 + b\phi + c, \quad a \neq 0, \tag{7}$$

and q_i $(i = 0, ..., m)$, a, b and c are real constants.

2.1 The some simple solutions of Riccati and CD equations

We know that Eq. (7) admits the following solutions:

Case 1 *When* $\Delta = b^2 - 4ac > 0$,

$$\phi_1 = -\frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\frac{1 + e^{\sqrt{\Delta}\xi}}{1 - e^{\sqrt{\Delta}\xi}}, \tag{8}$$

Case 2 *When* $\Delta = b^2 - 4ac = 0$,

$$\phi_2 = -\frac{b}{2a} - \frac{1}{a\xi}, \tag{9}$$

Case 3 *When* $\Delta = b^2 - 4ac < 0$,

$$\phi_3 = -\frac{b}{2a} + \frac{\sqrt{-\Delta}}{2a}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right), \quad \left|\frac{\sqrt{-\Delta}}{2}\xi\right| < \frac{\pi}{2}, \tag{10}$$

or

$$\phi_4 = -\frac{b}{2a} - \frac{\sqrt{-\Delta}}{2a}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right), \quad 0 < \frac{\sqrt{-\Delta}}{2}\xi < \pi. \tag{11}$$

It is easy to find that $m = 1$ by balancing the terms $(U')^2$ and U''' in Eq. (5). So we choose

$$U = q_0 + q_1\phi. \tag{12}$$

From Eqs. (7) and (12), we obtain

$$\begin{aligned} U' &= q_1a\phi^2 + q_1b\phi + q_1c \\ U'' &= 2q_1a^2\phi^3 + 3q_1ab\phi^2 + 2q_1ac\phi + q_1b^2\phi + q_1bc \\ U''' &= 6q_1a^3\phi^4 + 12q_1a^2b\phi^3 + 8q_1a^2c\phi^2 + 7q_1ab^2\phi^2 \\ &\quad +8q_1abc\phi + q_1b^3\phi + 2q_1ac^2 + q_1b^2c \end{aligned} \tag{13}$$

Substituting the Eq. (13) into Eq. (5), collecting all terms with the power in ϕ^i $(i = 0,1,2,3,4)$ and setting each of the obtained coefficients of ϕ^i to zero yields the following set of algebraic equations with respect to q_0,q_1 and β :

$$\begin{aligned} &6q_1a^3 - 3q_1^2a^2 = 0, \\ &-6q_1^2ab + 12q_1a^2b = 0, \\ &-\beta q_1a - 6q_1^2ac - 3q_1^2b^2 + 8q_1a^2c + 7q_1ab^2 = 0, \\ &-\beta q_1b + q_1b^3 - 6q_1^2bc + 8q_1abc = 0, \\ &-3q_1^2c^2 - \beta q_1c + 2q_1ac^2 + q_1b^2c = 0. \end{aligned} \tag{14}$$

for which we get the following solution:

$$\begin{aligned} q_1 &= 2a, \\ \beta &= b^2 - 4ac, \end{aligned} \tag{15}$$

where q_0 is arbitrarily chosen. Pay to attention that $\Delta = \beta$.

Therefore, from Eq. (12) and Cases 1-3, we obtained many families of exact travelling wave solution of CD equation (1)

Family 1: *Exponential solutions*

$$u_1 = q_0 + 2a\left(-\frac{b}{2a} + \frac{\sqrt{\beta}}{2a}\frac{1 + e^{\sqrt{\beta}(x+y-\beta t)}}{1 - e^{\sqrt{\beta}(x+y-\beta t)}}\right). \tag{16}$$

Family 2: *Rational solutions*

$$u_2 = q_0 + \frac{2}{x+y} \tag{17}$$

Family 3: *Periodic solutions*

$$u_3 = q_0 + 2a\left(-\frac{b}{2a} + \frac{\sqrt{-\beta}}{2a}\tan\left(\frac{\sqrt{-\beta}}{2}(x+y-\beta t)\right)\right), \tag{18}$$

where $\left|\frac{\sqrt{-\beta}}{2}(x+y-\beta t)\right| < \frac{\pi}{2}$, or

$$u_4 \;=\; q_0 + 2a\left(-\frac{b}{2a} - \frac{\sqrt{-\beta}}{2a}\cot\left(\frac{\sqrt{-\beta}}{2}(x+y-\beta t)\right)\right), \tag{19}$$

where $0 < \frac{\sqrt{-\beta}}{2}(x+y-\beta t) < \pi$.

2.2 The exact solutions of Eq. (5) by using the sinh-cosh solutions form of Riccati equation

We assume that the Riccati equation (7) has solutions in the following form

$$\phi = \sum_{i=1}^{m} \sinh^{i-1} w(A_i \sinh w + B_i \cosh w) + A_0, \tag{20}$$

where $dw/d\xi = \sinh w$ or $dw/d\xi = \cosh w$. It is easy to find that $m = 1$ by balancing ϕ' and ϕ^2. So we choose

$$\phi = A_0 + A_1 \sinh w + B_1 \cosh w \tag{21}$$

When $dw/d\xi = \sinh w$, we substitute Eq. (21) and $dw/d\xi = \sinh w$ into Eq. (7) and set the coefficient of $\sinh^i w \cosh^j w$, $(i = 0, 1, 2; j = 0, 1)$ to be zero. A set of algebraic equations is then obtained as

$$\begin{aligned} &a(A_1^2 + B_1^2) = B_1,\\ &2aA_1B_1 = A_1,\\ &2aA_0A_1 + bA_1 = 0,\\ &2aA_0B_1 + bB_1 = 0,\\ &a(A_0^2 + B_1^2) + bA_0 + c = 0, \end{aligned} \tag{22}$$

for which, we have the following solutions:

$$A_0 = -\frac{b}{2a}, \quad A_1 = 0, \quad B_1 = \frac{1}{a}, \quad c = \frac{b^2-4}{4a}, \quad a \neq 0 \tag{23}$$

or

$$A_0 = -\frac{b}{2a}, \quad A_1 = \pm\frac{1}{2a}, \quad B_1 = \frac{1}{2a}, \quad c = \frac{b^2-1}{4a}, \quad a \neq 0 \tag{24}$$

For $dw/d\xi = \sinh w$, we use separation of variables method to obtain the solutions

$$sinhw = -csc\xi, coshw = -coth\xi. \tag{25}$$

Therefore from (23),(24) and (21) we obtain

$$\phi_5 = -\frac{b + 2\coth\xi}{2a}, \tag{26}$$

or

$$\phi_{6,7} = -\frac{b \pm csch\xi + coth\xi}{2a}, \tag{27}$$

From Eqs. (26), (27) and (6), we obtain the corresponding soliton solutions of CD equation (1) as

$$u_5 = q_0 - b - 2\coth(x+y-\beta t), \tag{28}$$

and

$$u_{6,7} = q_0 - b \pm csch(x+y-\beta t) + coth(x+y-\beta t). \tag{29}$$

Similarly, when $dw/d\xi = \cosh w$, we obtain the following periodic solutions of CD equation (1):

$$u_8 = q_0 - b - 2\cot(x+y-\beta t), \quad 0 < x+y-\beta t < \pi, \tag{30}$$

and

$$u_{9,10} = q_0 - b \pm \csc(x+y-\beta t) + \cot(x+y-\beta t), \quad 0 < x+y-\beta t < \pi. \tag{31}$$

3 Exact and explicit traveling wave solutions for the PKP equation

Now, let us construct exact solutions for a physically important PKP equation (2). We take the following general transformation:

$$u = U(\xi), \qquad \xi = x + y - \beta t$$

we get, inserting the transformations above in Eq.(2),

$$-\beta U'' + \frac{3}{2}U'U'' + \frac{1}{4}U^{(4)} + \frac{3}{4}U'' = 0. \tag{32}$$

Integrating Eq. (32) once and choosing the constants of integration to be zero, we find

$$-\beta U' + \frac{3}{4}\left(U'\right)^2 + \frac{1}{4}U''' + \frac{3}{4}U' = 0. \tag{33}$$

3.1 The some simple solutions of Riccati and PKP equations

It is now straightforward to obtain $m = 1$ by balancing the terms $(U')^2$ and U''' in Eq. (33). Hence

$$U = q_0 + q_1\phi. \tag{34}$$

Substituting the Eq. (13) into Eq. (33), collecting all terms with the power in ϕ^i $(i = 0, 1, 2, 3, 4)$ and setting each of the obtained coefficients of ϕ^i to zero yields the following set of algebraic equations with respect to q_0,q_1 and β :

$$\begin{aligned}
&\frac{3}{2}q_1a^3 + \frac{3}{4}q_1^2a^2 = 0,\\
&\frac{3}{2}q_1^2ab + 3q_1a^2b = 0,\\
&-\beta q_1a + \frac{3}{2}q_1^2ac + \frac{3}{4}q_1^2b^2 + 2q_1a^2c + \frac{7}{4}q_1ab^2 + \frac{3}{4}q_1a = 0,\\
&\frac{1}{4}q_1b^3 + \frac{3}{2}q_1^2bc - \beta q_1b + \frac{3}{4}q_1b + 2q_1abc = 0,\\
&\frac{1}{4}q_1^2c^2 + \frac{1}{2}q_1ac^2 + \frac{1}{4}q_1b^2c - \beta q_1c + \frac{3}{4}q_1 = 0.
\end{aligned} \tag{35}$$

for which we get the following solution:

$$q_1 = -2a, \beta = -ac + \frac{1}{4}b^2 + \frac{3}{4} \tag{36}$$

and q_0 is arbitrarily chosen. Pay to attention that $\Delta = 4\beta - 3$.

Therefore, from Eq. (34) and Cases 1-3, we obtained many families of exact trvelling wave solutions of PKP equation (2):

Family 1: *Exponential solutions*

$$u_1 = q_0 - 2a\left(-\frac{b}{2a} + \frac{\sqrt{4\beta-3}}{2a}\frac{1 + e^{\sqrt{4\beta-3}(x+y-\beta t)}}{1 - e^{\sqrt{4\beta-3}(x+y-\beta t)}}\right). \tag{37}$$

Family 2: *Rational solutions*

$$u_2 = q_0 - \frac{2}{x + y - \frac{3}{4}t}. \tag{38}$$

Family 3: *Periodic solutions*

$$u_3 = q_0 - 2a\left(-\frac{b}{2a} + \frac{\sqrt{3-4\beta}}{2a}\tan\left(\frac{\sqrt{3-4\beta}}{2}(x+y-\beta t)\right)\right), \tag{39}$$

where $\left|\frac{\sqrt{3-4\beta}}{2}(x+y-\beta t)\right| < \frac{\pi}{2}$, or

$$u_4 = q_0 - 2a\left(-\frac{b}{2a} - \frac{\sqrt{3-4\beta}}{2a}\cot\left(\frac{\sqrt{3-4\beta}}{2}(x+y-\beta t)\right)\right), \tag{40}$$

where $0 < \frac{\sqrt{3-4\beta}}{2}(x+y-\beta t) < \pi$.

3.2 The exact solutions of Eq. (33) by using the sinh-cosh solutions form of Riccati equation

We assume that the Riccati equation (7) has solutions in the form (20). Therfore from Eqs. (26), (27) and (36), when $dw/d\xi = \sinh w$, we obtain the corresponding soliton solutions of PKP equation (2)

$$u_5 = q_0 + b + 2\coth(x+y-\beta t), \tag{41}$$

and

$$u_{6,7} = q_0 + b \pm csch(x+y-\beta t) + coth(x+y-\beta t), \tag{42}$$

respectively. Similarly, when $dw/d\xi = \cosh w$, we obtain the following periodic solutions of PKP equation (2):

$$u_8 = q_0 + b + 2\cot(x+y-\beta t), \quad 0 < x+y-\beta t < \pi, \tag{43}$$

and

$$u_{9,10} = q_0 + b \pm \csc(x+y-\beta t) + \cot(x+y-\beta t), \quad 0 < x+y-\beta t < \pi. \tag{44}$$

4 Conclusion

In this work we have used the homogeneous balance method to derive exact solutions with distinct physical structures. The method can also be applied to other nonlinear partial differential equations. The results revealed remarkable properities of the shapes in that the solutions may come as compactons, solitary pattern, periodic solutions, or solitons depending on the method used. Homogeneous balance method is not apporiate for obtaining compactons or solitary patterns solutions. The availability of mathematical computer software like *Mathematica* or *Maple* facilitates the tedious algebraic calculations. The method which we have proposed in this paper is also a standard, direct and computerizable method which allows us to do complicated and tedious algebraic calculations.

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