ISSN 1479-3889 (print), 1479-3897 (online)
International Journal of Nonlinear Science
Vol.3(2007) No.1,pp.3-7

Pseudo Orbit Tracing Property of Non-wandering Operator

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(Received 26 July 2006, accepted 7 September 2006)

Abstract: We introduce $\alpha-$ pseudo orbit and $\beta-$ traced in infinite dimensional separable Banach space.Then the pseudo orbit tracing property of invertible non-wandering operators is proved by applying the method of functional analysis, and this property is invariant under iterates of T.

Key words: non-wandering operator; invariant subspace; pseudo orbit; pseudo orbit tracing property

1 Introduction

It is well known that linear operators in finite-dimensional linear spaces can't be chaotic but the nonlinear operator may be. Only in infinite-dimensional linear spaces can linear operators have chaotic properties. This has attracted widely attention ([1]-[6]). In recent years, the orbits of operators as important tools in studying the dynamical properties of the operators and the invariant subspace theory ([1]) have received amazing achievements. For instance, cyclic operator, suphyperclic operator, hypercyclic operator, non-wandering operator and linear chaotic operator have been intensively studied recently ([1]-[6]). In the research field of dynamical system, the study of the orbit is also playing an important role. The concept of pseudo orbit firstly appeared in the work of Anosov ([7]). Since then the pseudo orbit and the pseudo orbit tracing property as important tools in studying dynamical systems are discussed by many researchers ([8]), however, the above works are restricted in finite dimensional compact metric spaces, and results in infinite dimensional spaces.

While in the research field of differential dynamical system, Axiom A dynamic system is an important subject. However,Axiom A dynamic system is restricted in finite dimensional compact Riemann space. As we know in infinite dimensional spaces, the tangent cluster at each point is infinite dimensional, so this case is much complicated. But due to the linear property of operators, its tangent bundle at each point is linear operator itself. On the basis of these results, Lixin Tian and other researchers introduced non-wandering operator in infinite-dimensional Banach space, which is the generalization of Axiom A dynamic system but different from it. They are new linear chaotic operators and relative to hypercyclic operators, but different from them([2]). In recent years, Jiangbo Zhou discussed the hereditayily hypercyclic decomposition of non-wandering operators in infinite dimensional Frechet space ([3]); Shaoguang Shi obtained non-wandering operator sequences on Banach space ([4]) and Lihong Ren studied n-multiple non-wandering operator([5]).

In order to discuss the stability of non-wandering operators, in this paper, we introduced the $\alpha-$ pseudo orbit and $\beta-$traced in infinite dimensional separable Banach space (see Definition 2.2). Then we proved non-wandering operator T has pseudo orbit tracing property relative to an infinite dimensional separable closed Banach space E which is different from the finite dimensional spaces (see Remark 3.5).

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The paper is organized as follows: Section 2 list the basic notations and definitions. Then in Section 3, the pseudo orbit tracing property of non-wandering operators and this property is invariant under iterates of T are proved.

2 Basic notation and definitions

Let $(X, \|\cdot\|)$ be an infinite dimensional separable $Banach$ space on real number field or complex number field K. Let $L(X)$ be the set of all bounded linear operators over X. We introduce the following notations. For $y \in X$, let

$$W_\eta^u(y) = \left\{ x \in E \left| \left\| T^k (y-x) \right\| > \eta, (k = 0, 1, 2 \cdots) \ \lim_{k \to +\infty} \left\| T^{-k} (y-x) \right\| = 0 \right. \right\}$$

$$W_\eta^s(y) = \left\{ x \in E \left| \left\| T^k (y-x) \right\| < \eta, (k = 0, 1, 2 \cdots) \ \lim_{k \to +\infty} \left\| T^{k} (y-x) \right\| = 0 \right. \right\}$$

$$W^u(y) = \left\{ x \in E \left| \lim_{k \to +\infty} \left\| T^{-k} (y-x) \right\| = 0 \right. \right\}$$

$$W^s(y) = \left\{ x \in E \left| \lim_{k \to +\infty} \left\| T^{k} (y-x) \right\| = 0 \right. \right\}$$

$$W^u(E) = \left\{ x \in X \left| \lim_{k \to +\infty} \left\| T^{-k} x - E \right\| = 0 \right. \right\}$$

$$W^s(E) = \left\{ x \in X \left| \lim_{k \to +\infty} \left\| T^{k} x - E \right\| = 0 \right. \right\}$$

Definition 2.1 ([2]). Let $(X, \|\cdot\|)$ be an infinite dimensional separable $Banach$ space. Suppose $T \in L(X)$.
(1) Assume that there exists a closed subspace $E \subset X$, which has hyperbolic structure:

$$E = E^u \oplus E^s, TE^u = E^u, TE^s = E^s,$$

where E^u, E^s are closed subspaces. In addition, there exist constants $\tau\,(0 < \tau < 1)$ and $C > 0$, such that for any $\xi \in E^u, k \in N$,

$$\left\| T^k \xi \right\| \geq C\tau^{-k} \|\xi\|$$

and for any $\eta \in E^s, k \in N$,

$$\left\| T^k \eta \right\| \leq C\tau^{k} \|\eta\|$$

(2) Assume also that $Per(T)$ is dense in$E$.

Then T is said to be a non-wandering operator relative to E.

Definition 2.2 Let T be a non-wandering operator relative to closed subspace $E \subset X$. Suppose that $\{x_i\}_{i=a}^b$ is a sequence in E and $\alpha > 0$.($a = -\infty$ or $b = +\infty$ is also allowed).If for each $i = a \cdots b-1$ we have $\|Tx_i - x_{i+1}\| < \alpha$, then $\{x_i\}_{i=a}^b$ is said to be a $\alpha-$pseudo orbit of T. For a given $\beta > 0$, if there is $y \in E$ such that $\left\| T^i y - x_i \right\| < \beta$ for each $i = a \cdots b$. Then we say that the $\alpha-$ pseudo orbit is $\beta-$ traced by the orbit sending from y. T is said to have the pseudo orbit tracing property. If for each $\beta > 0$, there is $\alpha = \alpha(\beta) > 0$ such that each $\alpha-$ pseudo orbit of T can be $\beta-$ traced by some point in E.

3 Pseudo orbit tracing property of non-wandering operator

Proposition 3.1 ([6]) Let T be a non-wandering operator relative to closed subspace $E \subset X$,and then for any $x \in W_\varepsilon^u(y)$ and $\varepsilon > 0$ small enough, $y \in E$. There exists $0 < \mu < 1$, such that

$$\left\|T^{-1}y - T^{-1}x\right\| \leq \mu \|y - x\|$$

for any $z \in W_\varepsilon^s(y)$ and $\varepsilon > 0$ small enough, $y \in E$. Then there exists $0 < \mu < 1$, such that

$$\|Tz - Tx\| \leq \mu \|z - x\|$$

Proposition 3.2 ([6]) For $\varepsilon > 0$ small enough, if $\|y - z\| < \delta < \varepsilon$, $y, z \in E$, then we have

$$(1) W_\varepsilon^u(y) \cap W_\varepsilon^s(z) \subset E \quad (2) W_\varepsilon^u(z) \cap W_\varepsilon^s(y) \subset E$$

Proposition 3.3 ([6]) Let T be a non-wandering operator relative to closed subspace $E \subset X$. Suppose $\beta > 0$ is a given constant and N is a given positive integer, then there exists $\alpha > 0$such that each $\alpha-$pseudo orbit $\{x_i\}_{i=0}^N \subset E$ can be $\beta-$traced by the orbit sending fromx_0.

Theorem3.4 *Let T be an invertible non-wandering operator relative to closed subspace $E \subset X$.Then T has pseudo orbit tracing property.*

Proof: Firstly, we consider $\alpha-$ pseudo orbit $\{x_i\}_{i=0}^N \subset E$, by Proposition 3.1. For $\eta > 0$ small enough, there exists $0 < \mu < 1$, such that, for any $y \in E$, we have $p \in W_\eta^u(y) \Rightarrow \left\|T^{-1}y - T^{-1}p\right\| \leq \mu \|y - p\|$ and $q \in W_\eta^s(y) \Rightarrow \|Ty - Tq\| \leq \mu \|y - q\|$.

Taking $\varepsilon > 0$ small enough such that $\varepsilon < \eta$, $\varepsilon\left(\frac{3}{2} + \frac{1}{1-\mu}\right) < \beta' < \beta$, by Proposition 3.2,for any $\varepsilon > 0$ small enough, if $\|y - z\| < \delta < \varepsilon$, $y, z \in E$, there exists $W_\varepsilon^u(y) \cap W_\varepsilon^s(z) \subset E$. Now, assuming N large enough, such that $\mu^N \varepsilon < \frac{\delta}{2}$, by Proposition 3.3, for given N and $\frac{\delta}{2} > 0$, there exists $\alpha > 0$, so that any $\alpha-$pseudo orbit $\{\mathrm{x}_i\}_{i=0}^N \subset E$ can be $\frac{\delta}{2}$-traced by the orbit sending from x_0, which is,

$$\left\|\mathrm{T}^i \mathrm{x}_o - \mathrm{x}_i\right\| < \frac{\delta}{2} \quad i = 0, 1, \cdots N \tag{3.1}$$

Since T is a invertible non-wandering operator in$E$, we have $T \in L(E), T^{-1} \in L(E)$. E is a closed Banach space, so T and T^{-1} are uniformly bounded. Let $\left\|T^j\right\| \leq M_1, \left\|T^{-j}\right\| \leq M_2$. In Proposition 3.3, letting $M = \max\{M_1, M_2\}$, and applying the method of Proposition 3.3, we obtain $\left\|T^{-i}x_i - x_0\right\| < \beta \quad i = 1, 2 \cdots N$. Therefore, corresponding to (3.1) , we can obtain

$$\left\|T^{-i}x_i - x_0\right\| < \frac{\delta}{2}, i = 0, 1, 2 \cdots N \tag{3.2}$$

(ii) Next, we only have to consider the case that either a or b is infinite. Other similar cases follow.
For $\alpha-$ pseudo orbit $\{\mathrm{x}_i\}_{\mathrm{i}=-\infty}^0$, we will prove that there exists some point x $\in$ E, such that $\left\|T^i x - x_i\right\| < \beta, i = 0, -1, -2 \cdots$.

Firstly,we prove that $\alpha-$pseudo orbit $\{x_j\}_{j=-rN}^0$ can be $\beta' - traced$ by the orbit sending from the point $\mathrm{y}_r \in$ E, so we only have to prove that there exists $\mathrm{y}_r \in$ E, such that $\left\|\mathrm{T}^j \mathrm{y}_r - \mathrm{x}_j\right\| < \beta'$, j $= 0, -1, -2 \cdots - rN, 0 < \beta' < \beta$.

By (3.2) we have

$$\left\|T^{-k}x_{jN} - x_{jN-k}\right\| < \frac{\delta}{2}, k = 0, 1, \cdots N \tag{3.3}$$

We define $\left\{x'_{-sN}\right\}_{s=0}^r$ via the following induction: Firstly, suppose$x'_0 = x_0$.

Letting $x'_{-sN} = W_\varepsilon^s\left(T^{-N}x'_{-(s-1)N}\right) \cap W_\varepsilon^u(x_{-sN})$, we have

$$\left\|\mathrm{T}^{-\mathrm{N}}\mathrm{x}'_{-(\mathrm{s}-1)\mathrm{N}} - \mathrm{x}'_{-\mathrm{sN}}\right\| < \varepsilon \tag{3.4}$$

$$\left\| x_{-sN} - x'_{-sN} \right\| < \varepsilon \tag{3.5}$$

From (3.3), (3.5) and Proposition 3.1, we obtain

$$\left\| T^{-N} x'_{-(s-1)N} - x_{-sN} \right\| < \mu^N \frac{\delta}{2} < \delta.$$

Therefore,from Proposition 3.2 we have $W_\varepsilon^s \left(T^{-N} x'_{-(s-1)N} \right) \cap W_\varepsilon^u \left(x_{-sN} \right) \neq \phi$, and the above-mentioned definition is meaningful.

Letting $\mathrm{y}_r = T^{rN} x'_{-rN}$, we will prove that $\{x_j\}_{j=-rN}^0$ can be $\beta' - traced$ by the orbit sending from the point $\mathrm{y}_r \in$ E.

Denote $j = (s-r)N - k$, $0 \le k \le N$, so we have

$$\left\| T^j \mathrm{y}_r - x_j \right\| = \left\| T^j T^{rN} x'_{-rN} - x_j \right\| = \left\| T^{sN-k} x'_{-rN} - x_{(s-r)N-k} \right\|$$

$$\le \left\| T^{sN-k} x'_{-rN} - T^{-k} x'_{(s-r)N} \right\| + \left\| T^{-k} x'_{(s-r)N} - T^{-k} x_{(s-r)N} \right\| + \left\| T^{-k} x_{(s-r)N} - x_{(s-r)N-k} \right\|$$

Since by (3.4) we get $\left\| x'_{-tN} - T^{-N} x'_{-(t-1)N} \right\| \le \varepsilon$, so using Proposition 3.1 we have

$$\left\| T^{N-k} x'_{-tN} - T^{-k} x'_{-(t-1)N} \right\| = \left\| T^{(N-k)} x'_{-tN} - T^{(N-k)} T^{-N} x'_{-(t-1)N} \right\| \le \mu^{N-k} \varepsilon$$

$$\left\| T^{sN-k} x'_{-tN} - T^{(s-1)N-k} x'_{-(t-1)N} \right\| \le \left\| T^{(sN-k)} x'_{-tN} - T^{(sN-k)} T^{-N} x'_{-(t-1)N} \right\| \le \mu^{sN-k} \varepsilon$$

Therefore, we have

$$\left\| T^{sN-k} x'_{-rN} - T^{-k} x'_{(s-r)N} \right\| \le \left\| T^{sN-k} x'_{-rN} - T^{(s-1)N-k} x'_{-(r-1)N} \right\| + \cdots\cdots$$

$$+ \left\| T^{N-k} x'_{-(r-s+1)N} - T^k x'_{-(r-s)N} \right\| \le \mu^{sN-k} \varepsilon + \cdots + \mu^{N-k} \varepsilon \le {}^{\varepsilon}\!/_{1-\mu}$$

By (3.5) and Proposition 3.1 we have $\left\| T^{-k} x'_{(s-r)N} - T^{-k} x_{(s-r)N} \right\| < \mu^k \varepsilon < \varepsilon$.

By (3.3) we have $\left\| T^{-k} x_{(s-r)N} - x_{(s-r)N-k} \right\| \le {}^{\delta}\!/_{2}$.

So we get

$$\left\| T^j \mathrm{y}_r - x_j \right\| \le \frac{1-\mu}{\varepsilon} + \varepsilon + \frac{\delta}{2} \le \left(\frac{1-\mu}{1} + \frac{3}{2} \right) \varepsilon < \beta' \quad \mathrm{j} = 0, -1, -2 \cdots, -rN \tag{3.6}$$

In addition, since $\mathrm{y}_r = T^{rN} x'_{-rN} \subset E$, we consider the sequence $\{y_r\}$, in fact , when $r \to +\infty$,$\{y_r\}$ is equivalent to $\{y_{r+n}\}$,$n \to +\infty$.By Proposition 3.1 and (3.4) we get

$$\begin{aligned} \left\| y_{r+n} - y_{r+n-1} \right\| &= \left\| T^{(r+n)N} x'_{-(r+n)N} - T^{(r+n-1)N} x'_{-(r+n-1)N} \right\| \\ &= \left\| T^{(r+n)N} \left(x'_{-(r+n)N} - T^{-N} x'_{-(r+n-1)N} \right) \right\| \\ &< \mu^{(r+n)N} \varepsilon \to 0, \quad n \to +\infty \end{aligned}$$

Hence, $\{y_r\}$ is a Cauchy sequence. Furthermore, E is a closed Banach space, so there exists $x \in E$, such that $y_r \to x$,$r \to +\infty$.

Therefore by (3.6) we have $\left\| T^i x - x_i \right\| = \underset{r \to +\infty}{limt} \left\| T^j y_r - x_j \right\| < \beta' < \beta \quad i = 0, -1, -2 \cdots$

For $\alpha-$ pseudo orbit $\{x_i\}_{i=0}^{+\infty}$, we consider $\alpha-$ pseudo orbit $\{x_{j+n}\}_{j=-\infty}^0 \quad n \in N$.

Through the above proof, we obtain that $\alpha-$ pseudo orbit $\{x_{j+n}\}_{j=-\infty}^0$ can be $\beta-$ traced by the orbit sending from the point $x_n \in E$, that is, $\left\| T^j x_n - x_{j+n} \right\| < \beta \quad j = 0, -1, -2 \cdots$.

Let $y = T^{-n} x_n \quad i = j + n$, and then we have

$$\left\| T^i y - x_i \right\| = \left\| T^{j+n} T^{-n} x_n - x_{j+n} \right\| = \left\| T^j x_n - x_{j+n} \right\| < \beta, i = 0, 1, 2 \cdots \quad .$$

Thus, for any $\alpha-$ pseudo orbit $\{x_i\}_{i=0}^{+\infty}$ can be $\beta-$traced by the orbit sending from the point $y = T^{-n}x_n$ $(n \in N)$.

Remark3.5 In finite dimensional spaces, considering the pseudo orbit tracing property, we always suppose that X is a compact metric space, but here we let E be an infinite dimensional separable closed Banach space. Since in Theorem 3.4, we can construct $\{y_r\}$, such that $\{y_r\}$ is a Cauchy sequence, then by the property of closed Banach space E does not to be compact.

Corollary3.6 If T is an invertible non-wandering operator relative to closed subspace $E \subset X$, T has pseudo orbit tracing property relative to E_i,where E_i are basic sets.

Proof: Via the construction of E_i(see [2]), we can get Corollary holds.

Corollary3.7 For any given $\varepsilon > 0, M > 0$, there exists $\alpha > 0$ such that a sequence $\{x_i\}_{i=-\infty}^{+\infty}$ is a $\alpha-$ pseudo orbit of T. Then for any $j \geq 0$, we have $\left\|T^M x_j - x_{M+j}\right\| \leq \varepsilon$.

Proof: Since $T \in L(E)$ and E is a closed Banach space, T is uniformly bounded. Thus, there exists $N > 0$ such that$\|T\| \leq N$. In addition, $\{x_i\}_{i=-\infty}^{+\infty}$ is a $\alpha-$pseudo orbit of T, so we have $\|Tx_i - x_{i+1}\| < \alpha$. Choose $\alpha = {}^{\varepsilon}\!/_{\max\left\{N^{M-1}, N^{M-2}, \cdots, N\right\}M}$. Thus we obtain

$$\begin{aligned}\left\|T^M x_j - x_{M+j}\right\| &\leq \left\|T^M x_j - T^{M-1}x_{j+1}\right\| + \left\|T^{M-1}x_{j+1} - T^{M-2}x_{j+2}\right\| + \cdots + \|Tx_{M+j-1} - x_{j+M}\| \\ &< N^{M-1}\alpha + N^{M-2}\alpha + \cdots + N\alpha + \alpha < \varepsilon\end{aligned}$$

Corollary3.8 If non-wandering operator T has pseudo orbit tracing property, so do T^m and T^{-m} for each $m \in N$.

Proof: Obviously, T^m and T^{-m} satisfy condition (1) in Definition 2.1. Periodic points of T are also the ones of T^m and T^{-m}. Since $Per\,(T)$is dense in$E$, Per(T^m) and Per(T^{-m}) are also dense in E. Therefore T^m and T^{-m} are also non-wandering operator relative to$E$. With the method of theorem 3.4 and corollary 3.7, T^m and T^{-m} also have pseudo orbit tracing property.

Acknowledgements

This research was supported by Outstanding Personnel Program in Six Fields of Jiangsu Province(NO.6-A-029) and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE,P.R.C.(NO.2002-383)

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