

Painleve Integrability, Backlund Transformation and Solitary Solutions' Stability of Modified DGH Equation

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Abstract:In this paper, modified DGH equation (named MDGH equation) is proved to be Painleve integrable. And the truncated Painleve expansion is used to obtain an auto- Backlund transformation . Finally, the solitary wave solutions of M DGH equation are proved to be orbital stable.

Key words: modified DGH equation; new solitary wave solution; Backlund transformation; orbital stable

1 Introduction

In 1993, Camassa and Holm derived the new completely integrable dispersive wave equation. For water waves using Hamiltonian methods, namely Camassa-Holm equation[1, 2, 3]:

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad (1.1)$$

Lixin Tian and other researchers[4] also considered generalized Camassa-Holm equation and the modified Camassa-Holm equation

$$u_t + ku_x - u_{xxt} + au^2u_x = 2u_x u_{xx} + uu_{xxx}$$

and obtained some new exact peaked solitary wave solutions.

Dullin Gottwald and Holm got an integrable shallow-water equation that combined the linear dispersion, namely DGH equation[5, 6, 7, 8, 9]

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x + \gamma u_{xxx} = 2u_x u_{xx} + uu_{xxx} \quad (1.2)$$

In this paper, we consider modified DGH equation(MDGH equation)

$$u_t - u_{xxt} + 2\omega u_x + au^2u_x + \gamma u_{xxx} = 2u_x u_{xx} + uu_{xxx} \quad (1.3)$$

where a is an arbitrary constant.

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2 Painleve integrability of MDGH equation

As we know, an equation is called Painleve integrable when the equation has Painleve property which means the solutions to the equation are single valued about an arbitrary singular manifold. The Kruskal method [10] is used to prove the integrability of MDGH equation.

u in Eq.(1.3) is expanded by a local Laurent expansion in the neighborhood of singular manifold $\phi(x, t) = 0$, namely

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha} \quad (2.1)$$

Substituting the leading orders of the solutions to (1.3)

$$u = u_0 \phi^\alpha \quad (2.2)$$

into (1.3) for balancing the terms $u^2 u_x$ and $2u_x u_{xx} + uu_{xxx}$, we have

$$\alpha = -2, \quad u_0 = \frac{24}{a} \phi_x^2 \quad (2.3)$$

Substituting (2.1), (2.2), (2.3) into (1.3) yields the recursion relation of the expansion coefficients u_j , namely

$$(j+1)(j-6)(j-8)u_j = F_j(\phi_x, \phi_t, \dots, u_0, u_1, \dots, u_{j-1}) \quad (2.4)$$

where F_j is a function of u_0, u_1, \dots, u_{j-1} and the derivatives of ϕ . From (2.4), we know that the three required resonance conditions are

$$j = -1, 6, 8 \quad (2.5)$$

The resonance at $j = -1$ represents the arbitrariness of the singularity manifold $\phi(x, t) = 0$. So we only prove the existence of arbitrary functions in the other two cases $j = 6, 8$ for the integrability of MDGH equation, namely

$$F_6 = 0, \quad F_8 = 0 \quad (2.6)$$

According to the method of Kruskal

$$\phi = x + \psi(t) \quad (2.7)$$

where $\psi(t)$ is an arbitrary function of t , and substitute (2.1), (2.2), (2.3), (2.7) into (1.3), we can obtain the following

$$u_0 = \frac{24}{a} \quad (2.8)$$

$$u_1 = u_3 = u_5 = 0 \quad (2.9)$$

$$u_2 = -\frac{1}{3}\psi_t - \frac{1}{3}\gamma \quad (2.10)$$

$$u_4 = \frac{1}{20}\psi_t - \frac{w}{10} - \frac{a}{180}\psi_t^2 - \frac{a\gamma}{90}\psi_t - \frac{a\gamma^2}{180} \quad (2.11)$$

$$u_7 = \frac{13a}{5760}\psi_{tt} - \frac{a^2}{8640}\psi_t\psi_{tt} - \frac{a^2\gamma}{8640}\psi_{tt} \quad (2.12)$$

Using Painleve analysis, we can also obtain auto-Backlund transformation of Eq.(1.3),

$$u = \frac{24\phi_x^2}{a\phi^2} - \frac{24\phi_{x,x}}{a\phi} + u_2 \quad (2.13)$$

where ϕ satisfies the following five equations(because the formula is so long, just one part of it is shown here):

$$\begin{aligned} -2au_2u_{2,x}\phi_{x,x} - \phi_{x,x,t} + u_{2,x,x}\phi_{x,x} - 2\omega\phi_{x,x,x} - au_2^2\phi_{x,x,x} + 2u_{2,x}\phi_{x,x,x} + 2u_{2,x}\phi_{x,x,x,x} \\ + \phi_{x,x,x,t} - \gamma\phi_{x,x,x,x} + u_2\phi_{x,x,x,x} = 0 \\ \vdots \quad \vdots \quad \vdots \end{aligned} \tag{2.14}$$

3 Stability of solitary solutions

We assume that Eq.(1.3) has the traveling wave solution formed as

$$u(x, t) = \phi_c(\varepsilon), \varepsilon = x - ct \tag{3.1}$$

where $c > 0$ is the wave speed and Eq.(1.3) becomes

$$-c\phi'_c + c\phi'''_c = -a\phi_c^2\phi'_c + 2\phi'_c\phi''_c + \phi_c\phi'''_c \tag{3.2}$$

where ' means to take derivative. In [11], we have obtain abundant solitary solutions of (3.2).

The main purpose of this section is to show the orbital stability of the solitary solutions.

Theorem 3.1 *The solitary solution ϕ_c is orbital stable if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < T \leq \infty$ and $u \in C([0, T]; H^1(R))$ is a solution to (1.1) with*

$$\| u_0 - \phi_c \|_{H^1} \leq \delta; \text{ then } \inf_{\varepsilon \in R} \| u(\cdot, t) - \phi_c(\cdot, -\varepsilon) \|_{H^1} \leq \varepsilon, \text{ for every } t \in [0, T].$$

where $u(x, t)$ as a solution to (1.3) means that it satisfies the equation in the sense of distribution and the following quantities are conserved:

$$\frac{1}{2} \int u^2 + u_x^2 dx, \quad -\frac{1}{2} \int_R \frac{1}{6} u^4 + uu_x^2 + 2wu^2 - \gamma u_x^2 dx. \tag{3.3}$$

Consider a Hamiltonian system

$$\frac{du}{dt} = JE'(u(t)) \tag{3.4}$$

which are locally well-posed in a space X ; E is a functional (the energy); J is a skew-symmetric linear operator; that is

$$\langle Ju, v \rangle = - \langle u, Jv \rangle \quad \forall u, v \in D(J) \tag{3.5}$$

We assume that Eq. (3.4) is invariant under a representation $T(\cdot)$ of a group on X . The solitary wave is a solution to (3.4) as

$$u(t) = T(\eta t)\phi_\eta \tag{3.6}$$

Now we recall a theorem proved by Grillakis et al. in [12,13] , in order to prove the orbital stability of the solitary waves of Eq. (1.3).

Theorem 3.2 [12, 13]. *If satisfying assumptions (i)–(iv), the is orbital stable.*

(i) *There is a bounded linear operator $B : X \rightarrow X^*$ such that $B^* = B, Q(u) : X \rightarrow R$ is defined by $Q(u) = \frac{1}{2} \langle Bu, u \rangle$.*

(ii) *(Existence of solutions). For each $x_0 \in X, \exists t_0 > 0$ and a solution u of Eq. (3.4) in the interval $[0, t)$ such that $u(0) = u_0$, and*

$$E(u(t)) = E(u_0)Q(u(t)) = Q(u_0) \quad \text{for } t \in [0, t_0).$$

(iii) *(Existence of traveling waves). There exists real $\eta_1 < \eta_2$ and a mapping*

(a) $\eta \rightarrow \phi_\eta$ *from the open interval (η_1, η_2) into X which is C^1 such that for each $\eta \in (\eta_1, \eta_2)$,*

$$(b) E'(\phi_\eta) = \eta Q'(\phi_\eta)$$

$$(c) T'(0)\phi_\eta \neq 0$$

(d) For each $\eta \in (\eta_1, \eta_2)$, $H_\eta = E''(\phi_\eta) + \eta Q''(\phi_\eta)$ has exactly one negative simple eigenvalue and has its kernel spanned by $T'(0)\phi_\eta$ and the rest of its spectrum is positive and bounded away from zero.

Proof: Note that equation (1.3) can be written in Hamiltonian form as

$$\frac{du}{dt} = \partial_x(1 - \partial_x^2)^{-1} E'(u(t)), \quad (3.7)$$

where $E(u(t)) = -\frac{1}{2} \int_R \frac{a}{6} u^4 + uu_x^2 + 2\omega u^2 - \gamma u_x^2 dx$.

We can verify (a)–(c) by letting

$$B = 1 - \partial_x^2, \quad Q(u) = \frac{1}{2} \int_R (u^2 + u_x^2) dx,$$

and

$$T(ct)\phi_c = \phi_c(x - ct), \quad E'(\phi_c) + cQ(\phi_c) = 0.$$

So we only have to check assumption (d).

Since $H_c = E''(u(t)) + cQ''(u(t)) = -\partial_x((c + \gamma - u)\partial_x) - 2au^2 + 2u_{xx} + 2c - 4u$.

Suppose λ is the eigenvalue to H_c , with the corresponding eigenfunction v ; that is

$$H_c v = \lambda v$$

Then by Liouville transformation

$$y = \int_0^x \frac{1}{\sqrt{2(c + \gamma) - 2\phi_c(z)}} dz$$

and $\psi(y) = (2(c + \gamma) - 2\phi_c(z))^{\frac{1}{4}} v(x)$, we obtain an equation for $\psi(y)$ as

$$F_c \psi(y) = (-\partial_y^2 + p_c(y) + c - 2\omega)\psi(y) = \lambda \psi(y)$$

where $p_c(y) = -a\phi_c^2(x) + \frac{3}{4}\phi_c'' - \frac{(\phi_c'(x))^2}{16(c + \gamma - \phi_c(x))}$, and $p_c(y) \rightarrow 0$ as $|y| \rightarrow 0$.

By the property of ϕ_c and due to the spectral theorem [8], we have the property of F_c as

(I) essential spectrum $[c, \infty)$;

(II) only finite eigenvalues which are less than c ;

(III) the function to the n th eigenvalue (in increasing order) has $n-1$ zeros.

Note that the properties (I)–(III) were preserved by Liouville transformation, therefore the operator H_c has these properties.

On the other hand, it is computed directly that

$$H_c(\phi_c') = 0 \quad \phi_c'(0) = 0 \quad \phi_c'(x) \neq 0 \text{ for } x \neq 0$$

Thus assumption (iii) followed from the properties (I)–(III) directly.

Therefore, the orbit is stable by using Theorem 2.

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