Spectral Decomposition of a General Chebyshev Maps

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Abstract: In this paper, a spectral decomposition of the Frobenius-Perron operator for general Chebyshev polynomials is constructed of the first kind. And defined a suitable dual pair or rigged Hilbert space which provides mathematical meaning to the spectral decomposition is defined. The spectra of the even Chebyshev maps do not contain the odd powers of $1/m$ and the odd eigenfunctions are in the null space of the Frobenius-Perron operator. The eigenvalues in the decompositions are the resonances of power spectrum and have magnitudes less than one as in the case of the family of Tent maps.

Keywords: general Chebyshev maps; spectral decomposition; Koopman operator; Frobenius-Perron operator

1 Introduction

The probabilistic nonlocal approach to the study of chaotic dynamics [1] aims at the study of the Frobenius-Perron operator and the Koopman operators [2] which describe the evolution of probability densities and the observables of the system correspondingly. The probabilistic approach to dynamical systems has well-known advantages over the conventional topological approaches based on trajectories, namely:

1) The evolution law for the probability densities is linear even if the underlying dynamics is non-linear.
2) For unstable systems, trajectories, although existing mathematical concepts, are operationally unattainable due to intrinsic computational limitations, while the computation of the evolution of probabilities is stable and therefore determinable.
3) The spectral decomposition of the linear evolution operator, employing the methods of functional analysis, provides a new computational tool for the study of the evolution of dynamical systems [3].

The Frobenius-Perron operator $U$ is the dual of the Koopman operator $V$ defined ([2],[3]) for any dynamical system $S$ acting on the measurable space $X$ as follows:

$$VA(x) = A(S(x))$$

$$(U \rho | A) = (\rho | VA)$$

where $(\rho | A) = \int_X dv(x)\rho(x)A(x)$ is the expectation value of the observable $A$ in the density $\rho$ with respect to the reference measure $v$ on $X$. If $v$ is invariant measure then $U1 = 1$, where $1(x) = 1$ is the constant equilibrium density.

The spectral decomposition of the Frobenius-Perron operator is the dual of the spectral decomposition of the Koopman operator. The knowledge of the eigenvalues and the associated eigenfunctions of
the Frobenius-Perron operator allows us to answer effectively all relevant questions concerning the decay properties of the correlation functions, as well as the analyticity properties of the power spectra of chaotic dynamical systems. Furthermore, the spectral decomposition of the Frobenius-Perron operator amounts to an efficient solution to the prediction problem by means of probabilities [3] and can also provide a spectral condition of controllability for chaotic dynamical systems [4].

The Koopman operator for exact endomorphisms [5,6], noninvertible maps have positive Kololgorov-Sinai entropy, being a unilateral shift [3,6, 7], does not admit a spectral decomposition in the Hilbert space of square integrable functions. Spectral decompositions do exist in extensions of the eigenvalue problem to suitable dual pairs corresponding to Gelfand triplets or rigged Hilbert spaces ([8]-[10]). When the extended operator results from meromorphic extensions of the resolvent, the eigenvalues correspond to the resonances of the power spectrum discussed by Pollicott and Ruell ([11]-[14]). Such generalized eigenvectors and eigenvalues for exact systems have been obtained by several authors [7]. A sufficient condition for resonances to arise from a meromorphic continuation of the resolvent is that the Koopman operator is a Fredholm-Riesz operator [15].

We present here the spectral decompositions of a class of chaotic maps constructed from a general Chebyshev polynomials of the first kind.

The general Chebyshev maps $S_m$, $m = 2, 3, \cdots$, on the interval $[-2, 2)$ are defined by the formula (see figure.1 and 2)

$$S_m(x) = 2 \sin(m \arcsin \frac{x}{2}) \quad \text{for} \quad m = 2, 3, \ldots \quad (1.2)$$

The absolutely continuous invariant measure for all general Chebyshev maps is [2]

$$dv(x) = \frac{1}{\pi \sqrt{4 - x^2}} dx \quad (1.3)$$

The ergodic and mixing properties of Chebyshev maps were studied by Adler and Rivlin [16]. The decomposition of Chebyshev maps were studied by Bi Qiao and I.Antoniou [17]. This paper motivated the study of the recurrence, dispersion [18] and the statistical properties [19] of the iterates of a general Chebyshev maps. The general Chebyshev maps were shown to simulate random process [20, 21].

2 Construction of the spectral decomposition

The spectral decompositions of general Chebyshev maps can be obtained from the spectral decompositions of the family of Tent maps [22] through the well-known [2] topological equivalence of these transformations. The transformation $g : [0, 2) \to [0, 1)$ defined by
The eigenvalues are given by the formula
\[ \lambda_n = \frac{2m+2}{m} - 2n, \quad n = 0, 1, \ldots, \left[\frac{m-1}{2}\right]. \]

The eigenvectors \( v_n \) of \( V \) are defined by the Koopman formula (1.1) [2]:
\[ V f(x) = f(g(x)) \quad \text{for} \quad f \in L^2_{([0,1], dx)}. \]

The transformation \( G \) intertwines the Koopman operators \( V \) of a general Chebyshev maps with the Koopman operators \( V_T \) of the family of Tent maps \( V = GV_T G^{-1} \).

Proof:
\[ V f(x) = f(S_m(x)) = f(g^{-1}T_m g(x)) = G^{-1} f(T_m g(x)) = V_T G^{-1} f(g(x)) = GV_T G^{-1} f(x). \quad (2.2) \]

The intertwining transformation \( G \) can now be used in order to obtain the spectral decomposition of \( V \) from the spectral decomposition of \( V_T \).

As the result of the symmetry of the odd general Chebyshev maps, the Koopman operator transforms all functions over \([-2, 2]\) onto odd functions. Indeed for any odd map
\[ S(x) = -S(-x), \]
We have \( V f(x) = f(S(x)) = f(-S(-x)) = V(-f(-x)) \)
Therefore, the space of the odd functions on \([-2, 2]\), \( f(x) = f(-x) \) is mapped onto the space of the odd functions under the Koopman operator.

The spectral decomposition of the Koopman operator for the family of Tent maps has been obtained [22]
\[ V_T = \sum_{j=0}^{+\infty} z_j |\phi_j\rangle \langle \phi_j|, \quad (2.3) \]

The eigenvalues are given by the formula
\[ z_j = \frac{1}{m^{j+1}} \left\{ \left[ \frac{m-1}{2} \right] + 1 + (-1)^j \left( \left[ \frac{m-2}{2} \right] + 1 \right) \right\}, \]
where \( [x] \) is the integer part of \( x \).

For the even Tent maps, \( m = 2, 4, \ldots \), the eigenvalues are
\[ z_j = \begin{cases} \frac{1}{m^j}, & j \text{ even} \\ 0, & j \text{ odd} \end{cases} \]

For the odd Tent maps, \( m = 3, 5, \ldots \), the eigenvalues are
\[ z_j = \begin{cases} \frac{1}{m^j}, & j \text{ even} \\ \frac{1}{m^j}, & j \text{ odd} \end{cases} \quad (2.4) \]

The eigenvectors [22] of \( V_T \), for the even Tent maps, \( m = 2, 4, \ldots \), are given by
The meaning of the spectral decomposition (9) is inherited from the meaning of the spectral decomposition of the family of Tent maps (2.3) is to be understood as follows:

\[
\langle \rho | V_T A \rangle = \sum_{n=0}^{+\infty} z_n (\rho | \phi_n(x)) (\phi_n(x) | A)
\]

for any density \( \rho \) in the space \( P(x) \) of polynomials and any observable \( A \) in the anti-dual space \( \times p(x) \). The spectral decomposition of a general Chebyshev maps should therefore be understood in the sense:
\[(\rho | VsA) = \sum_{n=0}^{+\infty} z_n (\rho | \phi_n \left( \frac{2}{\pi} \arcsin \left( \frac{x}{2} \right) \right)) (\varphi_n \left( \frac{2}{\pi} \arcsin \left( \frac{x}{2} \right) \right) | A) \] (2.8)

for any density \( \rho \) in the space \( \times \rho((2/\pi) \arcsin(\frac{x}{2})) \), and any observable \( A \) in the anti-dual space \( \times \rho((2/\pi) \arcsin(\frac{x}{2})) \).

### 3 Concluding remarks

The spectra of the family of general Chebyshev maps are identical to the spectra of the family of Tent maps due to the spectral equivalence induced by the topological equivalence. The spectral of even general Chebyshev maps do not contain the odd powers of \( 1/m \) and are not degenerate. The spectral of odd general Chebyshev maps have degeneracy but no Jordan blocks [23]. As a result of the symmetry of the odd general Chebyshev maps, the Koopman operator transforms all functions onto even functions and all odd eigenfunctions of the Frobenius-Perron operator are in the null space of the Frobenius-Perron operator as in the case of the Tent maps. The eigenvalues of the Frobenius-Perron operator \( U_S \) are just the resonances of the power spectrum can be shown by the extension formula of the Fehlholm determinant as the case of the family of Tent maps [23]. Therefore, the eigenvalues \( z_j < 1, j = 1, 2, \ldots \), determine the decay rates of the correlation functions and the rate of approach to equilibrium \( f_0 \).

The admissible test function spaces of polynomials function of \( 2/\pi \arcsin(x/2) \) exclude the Dirac delta functions. This means that the trajectories \( \delta(y'-S_n y), n = 0, 1, 2, \ldots \) are excluded from the domain of the Frobenius-Perron operator. This remark goes beyond the prediction problem, as it also reflects the intrinsically probabilistic character of unstable dynamical systems. The spectral decomposition (2.8) has the property that the dynamical properties are reflected in the spectrum because the eigenvalues are the powers of the Lyapounov time.

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### References


