

## On Box Dimensions of Profile Curves of SPS

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**Abstract:** In this paper, we study the box dimensions of the profile curves of Star Product Surfaces (SPS). The relations of the box dimensions between the profile curves of SPS and the curves used for constructing the SPS is obtained.

**Key words:** star product surface; profile curve; box dimension

### 1 Introduction

A lot of work has been devoted to the study of the nonlinear science [1, 2, 3, 4]. Fractal is one of the focuses in nonlinear science. In this paper, we studied a family of fractal surfaces. There are rough and irregular surfaces everywhere in the nature. Generally, their fractal dimensions, such as box dimensions, are larger than their topological dimensions, and then they are called fractal surfaces. Many literatures have been involved in studying and developing the methods for simulating these surfaces by using mathematical fractal surfaces [5, 6]. One of kinds of the most important and simplest fractal surfaces is Cartesian product fractal surfaces. Let  $A \in R^2$  is a fractal curve, and  $B = [a, b] \in R$  is a straight line segment. Moving  $A$  along  $B$  perpendicularly, we construct a fractal surface, which is Cartesian product fractal surface of  $A$  and  $B$ , noted by  $A \times B$ . It is showed[7], for any sets  $E \in R^n$  and  $F \in R^m$ ,

$$\overline{\dim}_B(E \times F) \leq \overline{\dim}_B E + \overline{\dim}_B F, \tag{1}$$

where  $E \times F$  is Cartesian product of  $E$  and  $F$ , and  $\overline{\dim}_B S$  is the upper box dimension of set  $S$ . Consequently, Cartesian product fractal surface  $A \times B$  has upper box dimension satisfying  $\overline{\dim}_B(A \times B) \leq \overline{\dim}_B A + 1$ . This kind of surfaces are often used to model natural rough surfaces, whereas they exhibit fractal feature only in one direction and cannot fit natural surface perfectly. In [8], authors improved this model by supposing that both of  $A$  and  $B$  are continuous curves.

**Definition 1.1.** [8] In 3-dimension Euclidian space  $R^3$ , let  $A : z = f(x), x \in [a, b]$ ,  $B : z = g(y), y \in [c, d]$  and they are continuous curves on the coordinate planes  $xoz$  and  $yozy$  respectively. In the space  $oxyz$ , the surface

$$F = \{(x, y, z) | x \in [a, b], y \in [c, d], z = f(x) + g(y)\} \tag{2}$$

is called star product surface of continuous curves  $A$  and  $B$ , or simply Star Product Surface (SPS), denoted by  $A * B$ .

Obviously, the SPS can be obtained by moving the continuous curve  $A$  ( $B$ ) along the continuous curve  $B$  ( $A$ ) orthogonally, if neglecting the location of the surface. Let curve  $A$  move along curve  $B$ , then a surface, which is so-called the Star Product Surface (SPS) of two curves  $A$  and  $B$ , denoted by  $A * B$ , can be constructed.

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Box dimension is one of the most important parameter used to quantitatively characterize rough and irregular objects. In [8], the box dimension of the SPS was studied. The relation of the box dimensions between the SPS and the curves which are used to construct the SPS are strictly proved, i.e.,

$$\dim_B(A * B) = \max\{\dim_B A, \dim_B B\}. \quad (3)$$

In this paper, we discuss the profiles of SPS, and obtained the relations of dimensions between arbitrary profile of SPS and the curves used to construct the SPS.

## 2 Some Lemmas

Let  $F$  be a subset of  $R^n$ , the lower and upper box dimensions are defined as:

$$\underline{\dim}_B F = \underline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}, \quad (4)$$

$$\overline{\dim}_B F = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}, \quad (5)$$

respectively. If they are equal, we refer to the common value as the box dimension of  $F$ , denoting

$$\dim_B F = \lim_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta}. \quad (6)$$

Box dimension is also called box counting dimension, Minkowski dimension, Bouligand dimension and so on. Where  $N_\delta(F)$  is taken to be any of the several values given in [7].

Now we add a item to the list of the values of  $N_\delta(F)$ . Suppose  $F$  be a subset of the Euclidean space  $R^n$  ( $n > 1$ ). For any integers  $i_1, i_2, \dots, i_{n-1}$ , let  $C_\delta(i_1, i_2, \dots, i_{n-1})$  be a column, i.e.,

$$\{(x, y) | x \in [(i_1 - 1)\delta, i_1\delta] \times [(i_2 - 1)\delta, i_2\delta] \times \dots \times [(i_{n-1} - 1)\delta, i_{n-1}\delta], y \in R\}.$$

Let  $\tilde{N}_\delta(F, i_1, i_2, \dots, i_{n-1})$  be the smallest number of the cubes of side  $\delta$  in  $C_\delta(i_1, i_2, \dots, i_{n-1})$  needed to cover  $F \cap C_\delta(i_1, i_2, \dots, i_{n-1})$ , and denote

$$\tilde{N}_\delta(F) = \sum_{i_1, i_2, \dots, i_{n-1}} \tilde{N}_\delta(F, i_1, i_2, \dots, i_{n-1}). \quad (7)$$

We call  $\tilde{N}(F)$  the smallest number of  $\delta$ -collum cubes that cover  $F$ .

**Lemma 2.1.** Define  $\tilde{N}_\delta(F)$  as described above. Then

$$\underline{\dim}_B F = \underline{\lim}_{\delta \rightarrow 0} \frac{\log \tilde{N}_\delta(F)}{-\log \delta} \quad (8)$$

$$\overline{\dim}_B F = \overline{\lim}_{\delta \rightarrow 0} \frac{\log \tilde{N}_\delta(F)}{-\log \delta}. \quad (9)$$

*Proof.* Let  $N_\delta(F)$  be the smallest number of  $\delta$ -net cubes in  $R^n$  that cover  $F$ . Then

$$N_\delta(F) \leq \tilde{N}_\delta(F) \leq 2N_\delta(F). \quad (10)$$

Therefore

$$\underline{\dim}_B F = \underline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta} = \underline{\lim}_{\delta \rightarrow 0} \frac{\log \tilde{N}_\delta(F)}{-\log \delta}$$

and

$$\overline{\dim}_B F = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_\delta(F)}{-\log \delta} = \overline{\lim}_{\delta \rightarrow 0} \frac{\log \tilde{N}_\delta(F)}{-\log \delta}.$$

The proof of this lemma is completed.  $\square$

According to this lemma, we can add  $\tilde{N}_\delta(F)$  to the list of  $N_\delta(F)$  used to calculate the box dimensions. For convenience, we will still use  $N_\delta(F)$  to denote  $\tilde{N}_\delta(F)$ , the smallest number of  $\delta$ -collum cubes that cover  $F$ , below in this paper except in the proof of Lemma 2.3.

Let  $E \subset R^n$ ,  $Z = F(X)$  is a function defined on  $E$ . The graph of this function is denoted by  $\Gamma(F, E)$ , i.e.,

$$\Gamma(F, E) = \{(X, Z) : Z = F(X), X \in E\}.$$

Obviously,  $\Gamma(F, E)$  is a subset of  $R^{n+1}$ .

**Lemma 2.2.** Let  $z = f(x)$ ,  $x \in [a, b]$  and  $z = g(y)$ ,  $y \in [c, d]$  be two functions, and denote  $A = \Gamma(f, [a, b])$ ,  $B = \Gamma(g, [c, d])$  respectively. If  $\dim_B A$  and  $\dim_B B$  exist, then

$$\lim_{\delta \rightarrow 0^+} \frac{\log[N_\delta(A) + N_\delta(B)]}{-\log \delta} = \max\{\dim_B A, \dim_B B\}. \quad (11)$$

And additionally, if  $\dim_B A > \dim_B B$ , then

$$\lim_{\delta \rightarrow 0^+} \frac{\log[N_\delta(A) - N_\delta(B) - (d-c)\delta^{-1}]}{-\log \delta} = \dim_B A. \quad (12)$$

*Proof.* Eq.(11) was proved in [8], so we only need to prove Eq.(12). Denote  $\lim_{\delta \rightarrow 0^+} \frac{\log N_\delta(A)}{-\log \delta} = D(A)$  and  $\lim_{\delta \rightarrow 0^+} \frac{\log N_\delta(B)}{-\log \delta} = D(B)$ . For any  $0 < \varepsilon < (D(A) - D(B))/2$ , there exists a  $0 < \delta_0 < 1$ , such that for any  $0 < \delta < \delta_0$ ,  $\frac{\log N_\delta(A)}{-\log \delta} > D(A) - \varepsilon$  and  $\frac{\log N_\delta(B)}{-\log \delta} < D(A) + \varepsilon$ , i.e.,  $N_\delta(A) > \delta^{-D(A)+\varepsilon}$  and  $N_\delta(B) < \delta^{-D(A)-\varepsilon}$ , then

$$N_\delta(A) - N_\delta(B) - (d-c)\delta^{-1} > \delta^{-D(A)+\varepsilon}[1 - \delta^{D(A)-D(B)-2\varepsilon} - (d-c)\delta^{D(A)-1-\varepsilon}].$$

Because  $D(A) - 1 - \varepsilon > D(A) - D(B) - 2\varepsilon > 0$ ,

$$\lim_{\delta \rightarrow 0^+} \frac{\log[N_\delta(A) - N_\delta(B) - (d-c)\delta^{-1}]}{-\log \delta} \geq D(A) - \varepsilon.$$

With arbitrariness of the  $\varepsilon$ , we have

$$\lim_{\delta \rightarrow 0^+} \frac{\log[N_\delta(A) - N_\delta(B) - (d-c)\delta^{-1}]}{-\log \delta} \geq D(A) = \dim_B A.$$

Because  $N_\delta(A) - N_\delta(B) - (d-c)\delta^{-1} < N_\delta(A) + N_\delta(B)$ , with Eq. (11), it is obvious that

$$\lim_{\delta \rightarrow 0^+} \frac{\log[N_\delta(A) - N_\delta(B) - (d-c)\delta^{-1}]}{-\log \delta} \leq \dim_B A.$$

Then the proof of the lemma is completed.  $\square$

It has been proved that Hausdorff dimension is invariant under bi-Lipchitz transformation. In fact, this result is also right for box dimension.

**Lemma 2.3.** [9] Let  $F$  be a subset of  $R^n$ , and  $\varphi$  be a bi-Lipschitz mapping on  $F$ , namely there are two constants  $C_1$  and  $C_2$  ( $0 < C_1 \leq C_2 < \infty$ ) such that for any  $x, y \in F$

$$C_1|x - y| \leq |\varphi(x) - \varphi(y)| \leq C_2|x - y|. \quad (13)$$

Then

$$\underline{\dim}_B F = \underline{\dim}_B \varphi(F), \quad (14)$$

$$\overline{\dim}_B F = \overline{\dim}_B \varphi(F), \quad (15)$$

where  $\varphi(F) = \{\varphi(x)|x \in F\}$ .

### 3 Main Results

Let  $A : z = f(x), x \in [a, b]$  and  $B : z = g(y), y \in [c, d]$  are continuous curves on the coordinate planes  $xoz$  and  $yozy$  respectively. And let  $A * B = \{(x, y, z) | x \in [a, b], y \in [c, d], z = f(x) + g(y)\}$  is a SPS of  $A$  and  $B$ . Now suppose plane  $\pi$  is perpendicular to the coordinate plane  $xoy$ , if  $(A * B) \cap \pi \neq \emptyset$ , generally they are curves. We will study the dimensions of these curves. In this paper, we only considered the plane  $\pi$  passing through the vertex  $(a, c, 0)$ . If the plane  $\pi$  intersects segments  $\{(x, d, 0) : x \in [a, b]\}$  (or  $\{(b, y, 0) : y \in [c, d]\}$ ) at  $(x_0, d, 0)$  (or  $(b, y_0, 0)$ ), the plane  $\pi$  is denoted by  $\pi(x_0, d)$  (or  $\pi(b, y_0)$ ). Obviously,  $\pi(b, d)$  passes through the diagonal of the  $[a, b] \times [c, d]$ . In the following section, we suppose that the box dimensions  $\dim_B A$  and  $\dim_B B$  both exist.

**Theorem 3.1.** *Let  $\Gamma(b, y_0) = (\pi(b, y_0)) \cap (A * B)$ ,  $\Gamma(x_0, d) = (\pi(x_0, d)) \cap (A * B)$ , namely  $\Gamma(b, y_0)$  and  $\Gamma(x_0, d)$  denote the profile curves of  $A * B$ , then*

$$\dim_B \Gamma(b, y_0) \leq \max\{\dim_B A, \dim_B B\}, \tag{16}$$

$$\dim_B \Gamma(x_0, d) \leq \max\{\dim_B A, \dim_B B\}, \tag{17}$$

where  $x_0 \in [a, b], y_0 \in [c, d]$ .

*Proof.* (1) Firstly, let  $a = c = 0, x_0 = y_0 = b = d = 1$ , namely  $A : z = f(x), x \in [0, 1], B : z = g(y), y \in [0, 1]$  and  $(\pi(1, 1)) \cap (A * B)$  is diagonal curve of  $A * B$ . Let columns

$$C_{\frac{1}{n}}(i, j) = I_i \times I_j \times \mathbb{R}, \quad i, j = 1, 2, \dots, n,$$

where  $I_k = [(k-1)/n, k/n]$ . Of course, the diagonal curve  $(\pi(1, 1)) \cap (A * B)$  is included in  $\bigcup_{1 \leq k \leq n} C_{\frac{1}{n}}(k, k)$ . On the other hand, define oscillation of the function  $h$  on  $E$

$$\text{osc}(h, E) = \sup\{|h(x) - h(y)| : x, y \in E\}.$$

For  $I_i = [\frac{i-1}{n}, \frac{i}{n}], i = 1, 2, \dots, n$ . Obviously,

$$\text{osc}(f(x) + g(y), I_i \times I_j) = \text{osc}(f, I_i) + \text{osc}(g, I_j), \tag{18}$$

and

$$N_{\frac{1}{n}}(A, i) = \left\lceil \frac{\text{osc}(f, I_i)}{n} \right\rceil, \quad N_{\frac{1}{n}}(B, j) = \left\lceil \frac{\text{osc}(g, I_j)}{n} \right\rceil,$$

$$N_{\frac{1}{n}}(A * B, i, j) = \left\lceil \frac{\text{osc}(f(x) + g(y), I_i \times I_j)}{n} \right\rceil,$$

where  $\lceil x \rceil$  is equal to the smallest integer not less than  $x$ . Because  $\lceil x + x^* \rceil \leq \lceil x \rceil + \lceil x^* \rceil$ , then  $N_{\frac{1}{n}}(A * B; k, k) \leq N_{\frac{1}{n}}(A; k) + N_{\frac{1}{n}}(B; k)$ . Therefore,  $N_{\frac{1}{n}}(\Gamma(1, 1)) \leq \sum_{k=1}^n N_{\frac{1}{n}}(A * B; k, k) \leq \sum_{k=1}^n N_{\frac{1}{n}}(A; k) + \sum_{k=1}^n N_{\frac{1}{n}}(B; k) = N_{\frac{1}{n}}(A) + N_{\frac{1}{n}}(B)$ . According to Lemma 2.2,

$$\dim_B \Gamma(1, 1) \leq \max\{\dim_B A, \dim_B B\}$$

(2) For general intervals  $[a, b], [c, d]$ , construct maps  $\phi_1 : R^2 \rightarrow R^2, (x, z) \mapsto (\frac{x-a}{b-a}, z)$ ,  $\phi_2 : R^2 \rightarrow R^2, (y, z) \mapsto (\frac{y-c}{d-c}, z)$ , and  $\phi : R^3 \rightarrow R^3, (x, y, z) \mapsto (\frac{x-a}{b-a}, \frac{y-c}{d-c}, z)$ . It is very easy to verify that  $\phi, \phi_1, \phi_2$  are bi-Lipschitz maps. And  $\phi_1(A) : z = f(a + (b-a)x), x \in [0, 1], \phi_2(B) : z = g(a + (b-a)y), y \in [0, 1], \phi(A * B) : z = f(a + (b-a)x) + g(a + (b-a)y), (x, y) \in [0, 1] \times [0, 1]$ , and  $\phi(\pi(b, d)) = \pi(1, 1)$ . Therefore, according to Lemma 2.3,  $\dim_B \Gamma(b, d) = \dim_B((A * B) \cap (\pi(b, d))) = \dim_B(\phi((A * B) \cap (\pi(b, d)))) = \dim_B(\phi(A * B) \cap \phi(\pi(b, d))) = \dim_B((\phi_1(A) * \phi_2(B)) \cap \pi(1, 1)) \leq \max\{\dim_B \phi(A), \dim_B \phi(B)\} = \max\{\dim_B A, \dim_B B\}$ .

(3) For  $\Gamma(b, y_0), y_0 \in [c, d]$ , let  $B' : z = g(y), y \in [c, y_0]$ , then  $B' \subseteq B$ , and  $\Gamma(b, y_0)$  is the diagonal curve of  $A * B'$ . Hence, according to (2),

$$\dim_B \Gamma(b, y_0) \leq \max\{\dim_B A, \dim_B B'\} \leq \max\{\dim_B A, \dim_B B\}.$$

Similarly, Eq.(17) can be proved. □

**Theorem 3.2.** Let  $x_0 \in [a, b]$ ,  $y_0 \in [c, d]$  and  $A[a, x_0] = \{(x, z) : z = f(x), x \in [a, x_0]\}$ ,  $B[c, y_0] = \{(y, z) : z = g(y), y \in [c, y_0]\}$ .

(1) If  $\dim_B A \neq \dim_B B[c, y_0]$ ,  $y_0 \in [c, d]$ , then

$$\dim_B \Gamma(b, y_0) = \max\{\dim_B A, \dim_B B[c, y_0]\}. \quad (19)$$

(2) If  $\dim_B A[a, x_0] \neq \dim_B B$ ,  $x_0 \in [a, b]$ , then

$$\dim_B \Gamma(x_0, d) = \max\{\dim_B A[a, x_0], \dim_B B\}. \quad (20)$$

*Proof.* Similar to the proof of Theorem 3.1, we still suppose firstly  $a = c = 0$ ,  $x_0 = y_0 = b = d = 1$ . Then the diagonal profile curve of SPS,  $\Gamma(1, 1) = (\pi(1, 1)) \cap (A * B)$  is plane curve  $z = f(t) + g(t)$ ,  $t \in [0, 1]$ . Now we suppose  $\dim_B A > \dim_B B$ , and use  $1/n$ -collum cubes (squares) in coordinate plane  $toz$  to cover  $\Gamma(1, 1)$ . In interval  $I_i = [(i-1)/n, i/n]$ ,  $\text{osc}((f+g)(x), I_i) \geq \text{osc}(f(x), I_i) - \text{osc}(g(x), I_i)$ . Because  $\lceil x - x^* \rceil \geq \lceil x \rceil - \lceil x^* \rceil - 1$ , then  $N_{\frac{1}{n}}(\Gamma(1, 1); I_i) \geq N_{\frac{1}{n}}(\Gamma(f, I_i)) - N_{\frac{1}{n}}(\Gamma(g, I_i)) - 1$ . So  $N_{\frac{1}{n}}(\Gamma(1, 1)) \geq N_{\frac{1}{n}}(A) - N_{\frac{1}{n}}(B) - n$ . It can be deduced that

$$\dim_B \Gamma(1, 1) \geq \lim_{n \rightarrow +\infty} \frac{\log(N_{\frac{1}{n}}(A) - N_{\frac{1}{n}}(B) - n)}{\log n} = \dim_B A. \quad (21)$$

Now let  $\phi_1 : [a, b] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$ ,  $(x, z) \mapsto (\frac{x-a}{b-a}, z)$ ;  $\phi_2 : [c, y_0] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$ ,  $(y, z) \mapsto (\frac{y-c}{y_0-c}, z)$ ;  $\phi : [a, b] \times [c, y_0] \times \mathbb{R} \rightarrow [0, 1] \times [0, 1] \times \mathbb{R}$ ,  $(x, y, z) \mapsto (\frac{x-a}{b-a}, \frac{y-c}{y_0-c}, z)$ . Because  $\phi_1, \phi_2, \phi$  are bi-Lipschitz transformation, according to Lemma 2.3 and inequality(21), we have  $\dim_B \Gamma(b, y_0) = \dim_B(\phi(A * B) \cap \phi(\pi(b, y_0))) = \dim_B((\phi_1(A) * \phi_2(B)) \cap \pi(1, 1)) \geq \max\{\dim_B \phi_1(A), \dim_B \phi_2(B)\} = \max\{\dim_B A, \dim_B B[c, y_0]\}$ . Then, by Theorem 3.1, (1) in Theorem 3.2 has been proved. Similarly, we can prove (2).  $\square$

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