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Semilinear Differential Equations with Nonlocal Conditions in Banach Spaces

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Abstract: In this paper we study the existence of mild solutions for the nonlocal Cauchy problem $x'(t) = Ax(t) + f(t, x(t)), 0 < t \leq b, x(0) = x_0$, by using the fixed point techniques, which extends and improves some existing results in this area.

Key words:measure of noncompactness; semilinear differential equation; fixed point; nonlocal condition; C_0-semigroup; mild solution

1 Introduction

In this paper we study the nonlocal initial value problem

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in (0, b], \tag{1.1}$$

$$x(0) = g(x), \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in Banach space X, $f : [0, b] \times X \to X$ and $g : C(0, b; X) \to X$.

The study of abstract nonlocal semilinear initial value problems was initiated by Byszewsk [5, 6, 7]. Among his several papers, he proves the existence and uniqueness of mild solutions when f and g satisfy Lipschitz type conditions. Subsequently, many authors are devoted to the study of nonlocal Cauchy problems because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Ntouyas and Tsamatos [15, 16], Byszewski and Akca [8], Liang, Liu and Xiao [12] study the case when $T(t)$ is compact and f, g satisfy appropriate conditions. Aizicovici [1, 2] studies the nonlocal Cauchy problems when A is a nonlinear m-accretive operator on X. Recently, Xue [20, 21] discusses the semilinear and nonlinear nonlocal problem (1.1), (1.2) by using the method of topological transformation, respectively, which avoids the difficulties associated with unbounded operators when $t = 0$. Other contributions see [4, 9, 11, 13, 19].

In this paper, we prove the existence results of mild solutions for (1.1), (1.2) without the compactness assumption on $T(t)$. The price that we pay to achieve this generalization, is that we have to strengthen the compactness hypothesis on f. But such hypothesis can be satisfied usually. So our work extends and improves many main results such as those in [8, 12, 15, 16, 20]. In addition, we emphasize that the proofs herein are different from the ones before. We try to make use of the properties of noncompact measures in proof, which also enables us to avoid the difficulties associated with unbounded operators when $t = 0$. Our basic tools are the methods and results for semilinear differential equations in Banach spaces, the properties of noncompact measures and fixed point techniques. In Section 2, we recall some definitions and facts about the measure of noncompactness and semilinear differential equations. In Section 3, we prove our main results.

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2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C(0, b; X)$ the space of X-valued continuous functions on $[0, b]$ with the norm $\|x\| = \sup\{\|x(t)\|, t \in [0, b]\}$ and by $L^1(0, b; X)$ the space of X-valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| \, \mathrm{d}t$.

Let us recall the following definitions.

Definition 2.1 *Let E^+ be the positive cone of an order Banach space $(E, \leq)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in E^+ is called a measure of noncompactness (MNC) on X if $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ for all bounded subsets $\Omega \subset X$, where $\overline{co}\Omega$ stands for the closed convex hull of Ω.*

The MNC Φ is said :

(i) $monotone$ if for all bounded subsets Ω_1, Ω_2 of X we have: $(\Omega_1 \subseteq \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))$;
(ii) $nonsingular$ if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for every $a \in X,\ \Omega \subseteq X$;
(iii) $regular$ if $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X.

One of the most important examples of MNC is the noncompactness measure of Hausdorff χ defined on each bounded subset Ω of X by

$$\chi(\Omega) = \inf\{\varepsilon > 0;\ \Omega\ has\ a\ finite\ \varepsilon - net\ in\ X\}.$$

It is well known that MNC χ enjoys the above properties (i)-(iii) and other properties (see [3, 10]).

Consider the Cauchy problem:

$$x'(t) = Ax(t) + \omega(t), \quad 0 < t \leq b, \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X (see [17]).

Definition 2.2 *A function $x \in C(0, b; X)$ is a mild solution to (2.1), (2.2) if:*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\omega(s) \, \mathrm{d}s$$

for all $t \in [0, b]$.

By a mild solution to (1.1), (1.2) we mean a function $x \in C(0, b; X)$ which satisfies

$$x(t) = T(t)g(x) + \int_0^t T(t-s)f(s, x(s)) \, \mathrm{d}s$$

for all $t \in [0, b]$.

It is well known that if $\omega \in L^1(0, b; X)$, then (2.1), (2.2) has a unique mild solution.

Definition 2.3 *A countable set $\{f_n\}_{n=1}^{+\infty} \subset L^1(0, b; X)$ is said to be semicompact if the sequence $\{f_n(t)\}_{n=1}^{+\infty}$ is compact in X for a.a. $t \in [0, b]$ and if there is a function $\mu \in L^1(0, T; R^+)$ satisfying $\sup_{n\geq 1} \|f_n(t)\| \leq \mu(t)$ for a.e. $t \in [0, b]$.*

Definition 2.4 *We call the operator $G : L^1(0, b; X) \to C(0, b; X)$ defined by*

$$Gf(t) = \int_0^t T(t-s)f(s) \, \mathrm{d}s, \ \ t \in [0, b], \tag{2.3}$$

as the Cauchy operator, where $T(t)$, $0 \leq t \leq b$, is the C_0-semigroup generated by A.

Now, we give the following properties about Cauchy operator G. (See [10] Theorem 4.2.2, Theorem 5.1.1, respectively.)

Proposition 2.1 *Let G be the Cauchy operator defined by (2.3), $\{f_n\}_{n=1}^{+\infty}$ a sequence of functions in $L^1(0,b;X)$. Assume that there exist μ,η in $L^1(0,b;R^+)$ satisfying*

$$\sup_{n\geq 1}\|f_n(t)\| \leq \mu(t) \ and\ \chi(\{f_n(t)\}_{n=1}^{+\infty}) \leq \eta(t)\ a.e.\ t\in[0,b].$$

Then for all $t\in[0,b]$, we have

$$\chi(\{(Gf_n)(t)\}_{n=1}^{+\infty}) \leq 2M\int_0^t \eta(s)\,\mathrm{d}s,$$

where M equals to $\sup_{0\leq t\leq b}\|T(t)\|$ and χ is the Hausdorff MNC.

Proposition 2.2 *Let G be the Cauchy operator defined by (2.3). Then for every semicompact set $\{f_n\}_{n=1}^{+\infty}\subset L^1(0,b;X)$ the set $\{Gf_n\}_{n=1}^{+\infty}$ is relatively compact in $C(0,b;X)$.*

The following fixed point theorem, a nonlinear alternative of Monch type, plays a key role in our existence of mild solutions for nonlocal Cauchy problem (1.1), (1.2). (see Theorem 2.2 in [14] or Theorem 3 in [18]).

Theorem 2.1 *Let E be a Banach space, U an open subset of E and $0\in U$. Suppose that $F:\overline{U}\to E$ is a continuous map which satisfies Monch's condition (that is, if $D\subseteq\overline{U}$ is countable and $D\subseteq\overline{co}(\{0\}\cup F(D))$, then $\overline{D}$ is compact) and assume that*

$$x\neq\lambda F(x)\quad for\ \ x\in\partial U\ \ and\ \ \lambda\in(0,1)$$

holds. Then F has a fixed point in $\overline{U}$.

3 Main results

In this section, we give the existence of the mild solutions for nonlocal Cauchy problem (1.1), (1.2).

We first give the following assumptions:

(HA) The strongly continuous semigroup of bounded linear operators $T(t)$, $0\leq t\leq b$, generated by A is equicontinuous;

$(Hf1)$ $f:[0,b]\times X\to X$, for a.e. $t\in[0,b]$, the function $f(t,\cdot):X\to X$ is continuous and for all $x\in X$, the function $f(\cdot,x):[0,b]\to X$ is measurable;

$(Hf2)$ there exists a function $m\in L^1(0,b;R^+)$ and a nondecreasing continuous function $\Omega:R^+\to R^+$ such that

$$\|f(t,x)\|\leq m(t)\Omega(\|x\|)$$

for all $x\in X$ and $t\in[0,b]$;

$(Hf3)$ There exists a function $h\in L^1(0,b;R^+)$ such that for every bounded $D\subset X$,

$$\chi(f(t,D))\leq h(t)\chi(D)$$

for a.e. $t\in[0,b]$, where χ is the Hausdorff MNC.

(Hg) $g:C(0,b;X)\to X$ is a continuous compact map such that $\|g(x)\|\leq c\|x\|+d, \forall x\in C(0,b;X)$, for some positive constants c and d.

Theorem 3.1 *Assume that the conditions $(HA),(Hf1)-(Hf3),(Hg)$ are satisfied. Then the nonlocal Cauchy problem (1.1), (1.2) has at least one mild solution on $[0,b]$ provided that there exists a constant $N>0$ with*

$$\frac{(1-Mc)N}{Md+M\Omega(N)\|m\|_{L^1}}>1, \tag{3.1}$$

and that

$$2M\|h\|_{L^1}<1, \tag{3.2}$$

where M equals to $\sup_{0\leq t\leq b}\|T(t)\|$.

Proof: We consider the operator $R : C(0, b; X) \to C(0, b; X)$ defined by

$$(Rx)(t) = (R_1x)(t) + (R_2x)(t), \tag{3.3}$$

with

$$(R_1x)(t) = T(t)g(x), \tag{3.4}$$

$$(R_2x)(t) = \int_0^t T(t-s)f(s, x(s))\,\mathrm{d}s \tag{3.5}$$

for all $t \in [0, b]$.

It is easy to see that the fixed point of R is the mild solution of nonlocal Cauchy problem (1.1), (1.2). Subsequently, we will prove that R has a fixed point by using the Theorem 2.1.

Step1. The operator R is continuous on $C(0, b; X)$. For this purpose, we assume that $x_n \to x$ in $C(0, b; X)$. Then by $(Hf1)$ we have that

$$f(s, x_n(s)) \to f(s, x(s)), \quad (n \to +\infty), \ \forall s \in [0, b].$$

Since $\|f(s, x_n(s)) - f(s, x(s))\| \le 2\Omega(N)m(s)$ for some integer N, by $(Hf2)$, (Hg) and the dominated convergence theorem we have

$$\begin{aligned}\|Rx_n - Rx\| &\le M\|g(x_n) - g(x)\| + M\int_0^b \|f(s, x_n(s)) - f(s, x(s))\|\,\mathrm{d}s \\ &\to 0, \quad as\ n \to +\infty,\end{aligned}$$

i.e., R is continuous, where $M = \sup_{0\le t\le b}\|T(t)\|$.

Step2. The Monch's condition holds.

Suppose that $D \subseteq B_r$ is countable and $D \subseteq \overline{co}(\{0\} \cup R(D))$, we show that $\chi(D) = 0$, where B_r is the open ball of the radius r centered at the zero in $C(0, b; X)$ and χ is the Hausdorff MNC.

Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{+\infty}$. By using the condition (HA) and (Hg), we can easily verify that $\{Rx_n\}_{n=1}^{+\infty}$ is equicontinuous. So, $D \subseteq \overline{co}(\{0\} \cup R(D))$ is also equicontinuous.

Now, from (Hg), $(Hf3)$, Proposition 2.1 and properties of MNC χ, it follows that

$$\begin{aligned}\chi(\{Rx_n\}_{n=1}^{+\infty}) &\le \sup_{t\in[0,b]}(\chi(\{T(t)g(x_n)\}_{n=1}^{+\infty}) + \chi(\{\int_0^t T(t-s)f(s, x_n(s))\,\mathrm{d}s\}_{n=1}^{+\infty})) \\ &\le 2M\int_0^b h(s)\sup_{t\in[0,b]}\chi(\{x_n(t)\}_{n=1}^{+\infty})\,\mathrm{d}s \\ &= 2M\|h\|_{L^1}\chi(\{x_n\}_{n=1}^{+\infty}).\end{aligned}$$

Thus, we get that

$$\chi(D) \le \chi(\overline{co}(\{0\} \cup R(D))) = \chi(R(D)) \le 2M\|h\|_{L^1}\chi(D),$$

which implies that $\chi(D) = 0$, since the condition (3.2) holds.

Step3. Now let $\lambda \in (0, 1)$ and $x = \lambda R(x)$. Then for $t \in [0, b]$

$$x(t) = \lambda T(t)g(x) + \lambda\int_0^t T(t-s)f(s, x(s))\,\mathrm{d}s,$$

and one has

$$\begin{aligned}\|x(t)\| &\le Mc\|x\| + Md + M\int_0^b m(s)\Omega(\|x(s)\|)\,\mathrm{d}s \\ &\le Mc\|x\| + Md + M\Omega(\|x\|)\int_0^b m(s)\,\mathrm{d}s.\end{aligned}$$

Consequently

$$\frac{(1-Mc)\|x\|}{Md+M\Omega(\|x\|)\|m\|_{L^1}}\leq 1.$$

Then by (3.1) there exists N such that $\|x\|\neq N$. Set

$$U=\{x\in C(0,b;X):\|x\|<N\}.$$

From the choice of U there is no $x\in\partial U$ such that $x=\lambda R(x)$ for some $\lambda\in(0,1)$. Thus we get a fixed point of R in $\overline{U}$ due to the Theorem 2.1, which is a mild solution to (1.1), (1.2). This completes the proof.

Remark 3.1 *We note that if $Mc<1$ and f satisfies sublinear growth condition, then condition (3.1) is automatically satisfied.*

Now we suppose that
(Hg') $g:C(0,b;X)\to X$ is Lipschitz continuous with constant k.
To the proof of next result, we need the following simple fact about Hausdorff MNC χ.

Lemma 3.1 *If $D\subseteq C(0,b;X)$ be bounded, then we have*

$$\sup_{t\in[0,b]}\chi(D(t))\leq\chi(D).$$

Proof: For arbitrary $\epsilon>0$, there exists $D_i\subseteq C(0,b;X)$, $1\leq i\leq n$, such that $D=\cup_{i=1}^{n}D_i$ and

$$diam(D_i)\leq 2\chi(D)+2\epsilon,\quad i=1,\ 2,\ \cdots,\ n.$$

Now, we have $D(t)=\cup_{i=1}^{n}D_i(t)$ for each $t\in[0,b]$, and,

$$\|x(t)-y(t)\|\leq\|x-y\|\leq diam(D_i)$$

for $x(t),y(t)\in D_i(t)$. From the above two inequalities, it follows that

$$2\chi(D(t))\leq diam(D_i(t))\leq diam(D_i)\leq 2\chi(D)+2\epsilon.$$

By the arbitrariness of ϵ, we get that $\chi(D(t))\leq\chi(D)$. Therefore, we have

$$\sup_{t\in[0,b]}\chi(D(t))\leq\chi(D).$$

Theorem 3.2 *Assume that the conditions $(HA),(Hf1)-(Hf3),(Hg')$ are satisfied. Then the nonlocal Cauchy problem (1.1), (1.2) has at least one mild solution on $[0,b]$ provided that there exists a constant $N>0$ with*

$$\frac{(1-Mk)N}{M\|g(0)\|+M\Omega(N)\|m\|_{L^1}}>1,\tag{3.6}$$

and that

$$M(k+2\|h\|_{L^1})<1.\tag{3.7}$$

Proof: On account of Theorem 3.1, we can prove that operator R defined by (3.3) is continuous on $C(0,b;X)$.

We now prove that R satisfies the Monch's condition.

For this purpose, Let $D\subseteq B_r$ be countable and $D\subseteq\overline{co}(\{0\}\cup R(D))$. We show that $\chi(D)=0$.

Without loss of generality, we may suppose that $D=\{x_n\}_{n=1}^{+\infty}$. By using the condition (HA), we can easily verify that $\{R_2x_n\}_{n=1}^{+\infty}$ is equicontinuous. Moreover, $R_1:D\to C(0,b;X)$ is Lipschitz continuous with constant Mk due to the condition (Hg'). In fact, for $x,y\in D$, we have

$$\begin{aligned}\|R_1x-R_1y\| &= \sup_{t\in[0,b]}\|T(t)g(x)-T(t)g(y)\|\\ &\leq M\|g(x)-g(y)\|\\ &\leq Mk\|x-y\|.\end{aligned}$$

So, from Proposition 2.1, Lemma 3.1 and properties of MNC χ, it follows that

$$\begin{aligned}\chi(\{Rx_n\}_{n=1}^{+\infty}) &\leq \chi(\{R_1x_n\}_{n=1}^{+\infty}) + \chi(\{R_2x_n\}_{n=1}^{+\infty}) \\ &\leq Mk\chi(\{x_n\}_{n=1}^{+\infty}) + \sup_{t\in[0,b]} \chi(\{\int_0^t T(t-s)f(s,x_n(s))\,\mathrm{d}s\}_{n=1}^{+\infty}) \\ &\leq Mk\chi(\{x_n\}_{n=1}^{+\infty}) + 2M\int_0^b h(s) \sup_{t\in[0,b]} \chi(\{x_n(t)\}_{n=1}^{+\infty})\,\mathrm{d}s \\ &\leq Mk\chi(\{x_n\}_{n=1}^{+\infty}) + 2M\|h\|_{L^1}\chi(\{x_n\}_{n=1}^{+\infty}) \\ &= M(k+2\|h\|_{L^1})\chi(\{x_n\}_{n=1}^{+\infty}).\end{aligned}$$

Thus, we get that

$$\chi(D) \leq \chi(\overline{co}(\{0\}\cup R(D))) = \chi(R(D)) \leq M(k+2\|h\|_{L^1})\chi(D),$$

which implies that $\chi(D) = 0$, since the condition (3.7) holds.

Now, with analogous arguments as in the proof of Theorem 3.1, we can get an open ball U by the condition (3.6), and there is no $x \in \partial U$ such that $x = \lambda R(x)$ for some $\lambda \in (0,1)$. Thus we get a fixed point of R in $\overline{U}$ due to Theorem 2.1, which is a mild solution to (1.1), (1.2). This completes the proof.

Remark 3.2 *We note that if $Mk < 1$ and f satisfies sublinear growth condition, then condition (3.6) is automatically satisfied. In addition, we try to make use of the properties of MNC χ in proof, which enables us to avoid the difficulties associated with unbounded operators when $t = 0$.*

Finally, if we use another MNC, we will prove the result of Theorem 3.1 in the case there is no equicontinuity of the semigroup $T(t)$ and condition (3.2). It is very interesting.

Theorem 3.3 *Let A be the infinitesimal generator of a strongly continuous semigroup of bounded linear operator $T(t)$, $0 \leq t \leq b$, in X. Assume that the conditions $(Hf1)-(Hf3),(Hg)$ are satisfied. Then the nonlocal Cauchy problem (1.1), (1.2) has at least one mild solution on $[0,b]$ provided the condition (3.1) holds.*

Proof: On account of Theorem 3.1, we should only prove that the function $R: C(0,b;X) \to C(0,b;X)$ given by formula (3.3) satisfies the Monch's condition.

For this purpose, let $D \subseteq B_r$ be countable and $D \subseteq \overline{co}(\{0\}\cup R(D))$. We will prove that D is relatively compact.

In the sequel we will denote by Φ the following measure of noncompactness in $C(0,b;X)$ defined by (see [10])

$$\Phi(\Omega) = \max_{E\in\Delta(\Omega)}(\alpha(E), mod_C(E)) \tag{3.8}$$

for all bounded subsets Ω of $C(0,b;X)$.

Where:

$\Delta(\Omega)$ stands for the set of countable subsets of $\Omega \subset C(0,b;X)$;

α is the real MNC defined as

$$\alpha(E) = \sup_{t\in[0,b]} e^{-Lt}\chi(E(t))$$

$$with\ E(t) = \{x(t); x\in E\},\ t\in[0,b];$$

$mod_C(E)$ is the modulus of equicontinuity of the set of functions E given by the formula

$$mod_C(E) = \lim_{\delta\to 0}\sup_{x\in E}\max_{|t_1-t_2|\leq\delta}\|x(t_1)-x(t_2)\|\ ;$$

$L > 0$ is a constant that we shall appropriately choose.

It was proved in [10] that Φ is well defined (i.e. there is $E_0 \in \Delta(\Omega)$ which achieves the maximum in (3.8)) and is a monotone, nonsingular, regular MNC.

Let us choose a constant $L > 0$ such that

$$q \overset{\text{def}}{=} 2M \sup_{t\in[0,b]} \int_0^t e^{-L(t-s)} h(s)\,\mathrm{d}s < 1, \tag{3.9}$$

where M is the constant from Proposition 2.1 and h is the summable function of assumption $(Hf3)$.

From the regularity of Φ, it is enough to prove that $\Phi(D) = (0, 0)$. Since $\Phi(R(D))$ is a maximum, let $\{y_n\}_{n=1}^{+\infty} \subseteq R(D)$ be the denumerable set which achieves its maximum.

Of course, there exists a set $\{x_n\}_{n=1}^{+\infty} \subseteq D$ such that

$$\begin{aligned} y_n(t) = (Rx_n)(t) &= (R_1x_n)(t) + (R_2x_n)(t) \\ &= T(t)g(x_n) + \int_0^t T(t-s)f(s, x_n(s))\,\mathrm{d}s \end{aligned} \tag{3.10}$$

for all $n \geq 1$, $t \in [0, b]$.

Now, we give an estimate for $\alpha(\{y_n\}_{n=1}^{+\infty})$. By using condition $(Hf3)$, we have

$$\begin{aligned} \chi(\{f(s, x_n(s))\}_{n=1}^{+\infty}) &\leq h(s)\chi(\{x_n(s)\}_{n=1}^{+\infty}) \\ &\leq e^{Ls}h(s) \sup_{t\in[0,b]} e^{-Lt}\chi(\{x_n(t)\}_{n=1}^{+\infty}) \\ &= e^{Ls}h(s)\alpha(\{x_n\}_{n=1}^{+\infty}). \end{aligned} \tag{3.11}$$

From the Proposition 2.1 and (3.11), we get that

$$\begin{aligned} \chi(\{(R_2x_n)(t)\}_{n=1}^{+\infty}) \leq 2M \int_0^t e^{Ls}h(s)\alpha(\{x_n\}_{n=1}^{+\infty})\,\mathrm{d}s \\ = 2M\alpha(\{x_n\}_{n=1}^{+\infty}) \int_0^t e^{Ls}h(s)\,\mathrm{d}s. \end{aligned} \tag{3.12}$$

From (3.10), (3.11) and (3.12), it follows that

$$\begin{aligned} \alpha(\{y_n\}_{n=1}^{+\infty}) &= \sup_{t\in[0,b]} e^{-Lt}\chi(\{y_n(t)\}_{n=1}^{+\infty}) \\ &\leq \sup_{t\in[0,b]} e^{-Lt}\chi(\{(R_1x_n)(t) + (R_2x_n)(t)\}_{n=1}^{+\infty}) \\ &\leq \sup_{t\in[0,b]} e^{-Lt}2M\alpha(\{x_n\}_{n=1}^{+\infty}) \int_0^t e^{Ls}h(s)\,\mathrm{d}s \\ &= \alpha(\{x_n\}_{n=1}^{+\infty})2M \sup_{t\in[0,b]} \int_0^t e^{-L(t-s)}h(s)\,\mathrm{d}s \\ &= \alpha(\{x_n\}_{n=1}^{+\infty})q. \end{aligned} \tag{3.13}$$

Therefore, we have that

$$\alpha(\{x_n\}_{n=1}^{+\infty}) \leq \alpha(D) \leq \alpha(\overline{co}(\{0\} \cup R(D))) \leq \alpha(\{y_n\}_{n=1}^{+\infty}) \leq \alpha(\{x_n\}_{n=1}^{+\infty})q.$$

From (3.9), we obtain that

$$\alpha(\{x_n\}_{n=1}^{+\infty}) = \alpha(D) = \alpha(\{y_n\}_{n=1}^{+\infty}) = 0.$$

Coming back to the definition of α, we can see that

$$\chi(\{x_n(t)\}_{n=1}^{+\infty}) = \chi(\{y_n(t)\}_{n=1}^{+\infty}) = 0 \tag{3.14}$$

for every $t \in [0, b]$.

So, we obtain that $\{f(\cdot, x_n(\cdot))\}_{n=1}^{+\infty} \subset L^1(0, b; X)$ is semicompact. In fact, from (3.14) and $(Hf3)$, we have

$$\chi(\{f(t, x_n(t))\}_{n=1}^{+\infty}) \leq h(t)\chi(\{x_n(t)\}_{n=1}^{+\infty}) = 0,$$

i.e. $\{f(t, x_n(t))\}_{n=1}^{+\infty}$ is relatively compact for a.e. $t \in [0, b]$; Moreover, from the fact $\{x_n\}_{n=1}^{+\infty} \subseteq D \subseteq B_r$ and $(Hf2)$, we get

$$\|f(t, x_n(t))\| \leq m(t)\Omega(\|x_n(t)\|) \leq \Omega(r)m(t)$$

for a.e. $t \in [0, b]$ and every $n \geq 1$.

Further, we consider the Cauchy operator G defined by (2.3). It is easy to see that

$$R_2(\{x_n\}_{n=1}^{+\infty}) = G(\{f(\cdot, x_n(\cdot))\}_{n=1}^{+\infty}).$$

However, $G(\{f(\cdot, x_n(\cdot))\}_{n=1}^{+\infty})$ is relatively compact due to Proposition 2.2. So is the set $R_2(\{x_n\}_{n=1}^{+\infty})$.

On the other hand, by the strong continuity of $T(t)$ and the compactness of g, we can also easily verify that the set $R_1(\{x_n\}_{n=1}^{+\infty})$ is relatively compact.

Hence, from (3.10), we conclude that the set $\{y_n\}_{n=1}^{+\infty} \subset C(0, b; X)$ is relatively compact. Since Φ is a monotone, nonsingular, regular MNC, we have that

$$\Phi(D) \leq \Phi(\overline{co}(\{0\} \cup R(D))) \leq \Phi(R(D)) = \Phi(\{y_n\}_{n=1}^{+\infty}) = (0, 0).$$

Therefore, D is relatively compact. This completes the proof.

Remark 3.3 *In [8], the authors discuss a related semilinear nonlocal problem when g is convex and compact on a given ball. From the above theorem, however, we can see many key conditions in [8] are not required, such as the compactness of semigroup $T(t)$ and the convexity of g. And our boundedness conditions on f and g are weaker than theirs too.*

Remark 3.4 *In this paper, we require f to satisfy a compactness condition $(Hf3)$, but do not require the compactness of semigroup $T(t)$. Note that if f is compact or Lipschitz continuous, then condition $(Hf3)$ is satisfied. Therefore, our work extends and improves many previous results such as those in [8, 12, 15, 16, 20] and so on, where they need the compactness of $T(t)$ or f, or the Lipschitz continuity of f.*

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References

[1] S. Aizicovici, M. Mckibben: Existence results for a class of abstract nonlocal Cauchy problems: *Nonlinear Anal.,* **39** (2000) 649–668.

[2] S. Aizicovici, H. Lee: Nonlinear nonlocal Cauchy problems in Banach spaces, *Appl. Math. Lett.,* **18** (2005) 401–407.

[3] J. Banas, K. Goebel: "Measure of Noncompactness in Banach Spaces," *Lecture Notes in Pure and Appl. Math.,* **60,** Marcel Dekker, New York, 1980.

[4] M. Benchohra, S. Ntouyas: Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces, *J. Math. Anal. Appl.,* **258** (2001) 573–590.

[5] L. Byszewski: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.,* **162** (1991) 494–505.

[6] L. Byszewski: Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem, *Zesz. Nauk. Pol. Rzes. Mat. Fiz.,* **18** (1993) 109–112.

[7] L. Byszewski and V. Lakshmikantham: Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space, *Appl. Anal.,* **40** (1990) 11–19.

[8] L. Byszewski and H. Akca: Existence of solutions of a semilinear functional-differential evolution nonlocal problem, *Nonlinear Anal.* **34** (1998) 65–72.

[9] X. Fu and K. Ezzinbi: Existence of solutions for neutral functional differential evolution equations with nonlocal conditions, *Nonlinear Anal.* **54** (2003) 215–227.

[10] M. Kamenskii, V. Obukhovskii, P. Zecca: "Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces," *De Gruyter Ser. Nonlinear Anal. Appl.,* vol.**7,** de Gruyter, Berlin, 2001.

[11] J. Liang, J. Van Casteren, T.-J.Xiao: Nonlocal Cauchy problems for semilinear evolution equations, *Nonlinear Anal.* **50A** (2002) 173–189.

[12] J. Liang, J. Liu, T.-J.Xiao: Nonlocal Cauchy problems governed by compact operator families, *Nonlinear Anal.* **57** (2004) 183–189.

[13] Y. Lin and J. Liu: Semilinear integrodifferential equations with nonlocal Cauchy problems, *Nonlinear Anal.* **26** (1996) 1023–1033.

[14] H. Monch: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980) 985–999.

[15] S. Ntouyas, P. Tsamotas: Global existence for semilinear evolution equations with nonlocal conditions, *J. Math. Anal. Appl.,* **210** (1997) 679–687.

[16] S. Ntouyas, P. Tsamotas,:Global existence for semilinear integrodifferential equations with delay and nonlocal conditions, *Anal. Appl.,* **64** (1997) 99–105.

[17] A.Pazy:"Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.

[18] R. Precup: On the topological transversality principle, *Nonlinear Anal.* **20** (1993) 1–9.

[19] T. J. Xiao, J. Liang: Existence of classical solutions to nonautonomous nonlocal parabolic problems, *Nonlinear Anal.* **63** (2005) 225–232.

[20] X. Xue: Existence of solutions for semilinear nonlocal Cauchy problems in Banach spaces, *Elec. J. Diff. Equ.,* **64** (2005) 1–7.

[21] X. Xue: Nonlinear differential equations with nonlocal conditions in Banach spaces, *Nonlinear Anal.* **63** (2005) 575–586.