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On Adomian Decomposition Method for Solving Reaction Diffusion Equation

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Abstract: Adomian decomposition method (ADM) has been applied to solve many differential equations. In this paper we have used the Adomian decomposition method for solving nonlinear parabolic equation with variable parameters depends on time and space. We will show the existence of exact or analytic solution for a well-known reaction diffusion equation by using this method.

Keywords: Adomian decomposition method; reaction diffusion equation; nonlinear parabolic equation
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1 Introduction

Recently a great deal of interest has been focused on the applications of Adomian's decomposition method to solve a wide variety of stochastic and deterministic problems, see for a survey [1-3,9]. The Adomian's goal is to find a method to unity linear and nonlinear, ordinary and partial differential equations for solving initial and boundary value problems and in this paper we shall study a well known parabolic equation.

Various phenomenon of mathematical biology, engineering and physical problems can be described by linear or nonlinear parabolic equations. These linear or nonlinear models, as well as their analytic solutions, are of fundamental importance for applied sciences.

Our object in this paper is the desire to obtain exact solutions or analytic solutions to such linear and nonlinear parabolic equations with biological variable or physical parameters.

The Adomian decomposition scheme is a method for solving a wide range of problems whose mathematical models yield equation or system of ordinary or partial differential equations (see [1]) and the exact solution for heat equation is gained by using this method in [9].

In this paper we will use the "ADM" method to provide an analytic solution for the nonlinear initial-boundary value parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u + au(1-\frac{u}{N}) & (t,x)\in(0,T)\times\Omega, \\ u(t,x)=0 & (t,x)\in(0,T)\times\partial\Omega, \\ u(0,x)=u_0(x)\geq 0 & x\in\Omega \end{cases} \tag{1}$$

where $D > 0$ is the diffusion coefficient, $a > 0$ is the linear reproduction rate and $N > 0$ is the carrying capacity of the environment. (see Murry [8] for details).

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In many ecological systems, harvesting or predation of the species this model occurs. For example, fishing or hunting of the species u could happen. Mathematically, equation (1) generate a semiflow in the sobolev space $W_0^{1,2}(\Omega)$. In this logistic case, the dynamics of (1) has been completely studied by Henry in [6] and moreover it is often called Fisher's equation after Fisher [5], and it was also studied by Kolmogoroff et. al. in [7].

Here we want to solve this problem by using "ADM". The paper is organized as follows, in section 2 we present the analysis of the ADM applied to linear and nonlinear parabolic problem, in section 3 we apply this method for solving our problem and find an analytic solution.

2 Adomian decomposition method

Adomian (see [2],[3] for example) asserts that the decomposition method provides an efficient and computationally convenient method for generating approximate series solutions to a wide class of equations. In order that this paper will be reasonably self-contained, we describe here how this method is applied.

We consider problem (1) in the one dimension space e.g. $\Omega = [0, \pi]$ and $T = 1$ and so for the real value $u(t, x)$, the reaction diffusion equation has the following form

$$u_t = Du'' + au(1 - \frac{u}{N}) \tag{2}$$

with initial-boundary conditions

$$\begin{cases} u(t,0) = u(t,\pi) = 0 & t \in (0,1), \\ u(0,x) = u_0(x) \geq 0 & x \in [0,\pi]. \end{cases} \tag{3}$$

The equation (2) can be written as

$$L_t(u) = DL_x(u) + N(u) \tag{4}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial^2}{\partial x^2}$ and $N(u) = au(1 - \frac{u}{N})$.

If we operate the two sides of (4) with the inverse of the operator L_t, we have

$$u(t,x) = u(0,x) + DL_t^{-1}L_x(u) + L_t^{-1}N(u).$$

Substitute initial condition of (3) in the last formula we get

$$u(t,x) = u_0 + DL_t^{-1}L_x(u) + L_t^{-1}N(u). \tag{5}$$

To solve problem (2-3) by Adomian decomposition method, we work as usual in this procedure: we decompose the unknown function $u(t, x)$ by a series of components defined by

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x) \tag{6}$$

and a series of Adomian's polynomials for decomposition of nonlinear term $N(u)$ such as:

$$N(u) = \sum_{n=0}^{\infty} A_n(t,x). \tag{7}$$

The polynomials A_n are given by the following general algorithms

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}N(u_\lambda)]_{\lambda=0} \ , \ n = 0,1,2,\cdots \tag{8}$$

where we write

$$u_\lambda = \sum_{n=0}^{\infty} \lambda^i u_i.$$

For example

$$\begin{cases} A_0 = N(u_0) \\ A_1 = u_1 N'(u_0) \\ A_2 = u_2 N'(u_0) + \frac{1}{2} u_1^2 N''(u_0) \\ A_3 = u_3 N'(u_0) + u_1 u_2 u_2 N''(u_0) + \frac{1}{6} u_1^3 N'''(u_0) \\ \cdots\cdots \end{cases}$$

the remaining components $u_n(t,x)$ can be calculated as follows:

$$\begin{cases} u_1 = \int_0^t (Du_0'' + A_0)dt \\ u_2 = \int_0^t (Du_1'' + A_1)dt \\ \cdots \\ u_n = \int_0^t (Du_{n-1}'' + A_{n-1})dt. \end{cases}$$

So we can calculate the terms of $u(t,x) = \sum_{n=0}^{\infty} u_n(t,x)$ which is approximated by

$$\phi_m[u] = \sum_{n=0}^{m-1} u_n.$$

We can choose m such that the approximation $\phi_m[u]$ has desired accuracy, with more terms then we will have more accuracy.

In the next section we will apply this method for the problem 2-3.

3 Examples

Example 1. Consider the nonlinear parabolic problem (2-3) with initial condition $u(0,x) = 1 \geq 0$ on $[0,\pi]$, thus we have

$$\begin{aligned} u_0 &= 1 \\ u_1 &= \int_0^t (Du_0'' + A_0)dt = a(1 - \frac{1}{N})t \\ u_2 &= \int_0^t (Du_1'' + A_1)dt = a^2(1 - \frac{1}{N})(1 - \frac{2}{N})\frac{t^2}{2} \\ u_3 &= \int_0^t (Du_2'' + A_2)dt = [a^3(1 - \frac{1}{N})(1 - \frac{2}{N})^2 + a^2(1 - \frac{1}{N})^2]\frac{t^3}{6} \\ u_4 &= \int_0^t (Du_3'' + A_3)dt = [a^4(1 - \frac{1}{N})(1 - \frac{2}{N})^2]\frac{t^4}{24} + [a^4(1 - \frac{1}{N})^2(1 - \frac{2}{N})^2]\frac{t^4}{8} \end{aligned}$$

and so on, the solution $u(t,x)$ in the series form is given by

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x).$$

Example 2. Consider the nonlinear parabolic problem (2-3) with initial condition $u(0,x) = x \geq 0$ on

$[0,\pi]$, thus we have

$$\begin{aligned}
u_0 &= x\\
u_1 &= \int_0^t (Du_0'' + A_0)dt = ax(1-\frac{x}{N})t\\
u_2 &= \int_0^t (Du_1'' + A_1)dt = [D^2a(1-\frac{2}{N}) + a^2(1-\frac{2}{N})(1-\frac{x}{N})x^2]\frac{t^2}{2}\\
u_3 &= \int_0^t (Du_2'' + A_2)dt = [D^2a^2(1-\frac{2}{N})^2x + a^3(1-\frac{2}{N})^2(1-\frac{x}{N})x^3 + a^3(1-\frac{2}{N})x^2(1-\frac{x}{N})^2]\frac{t^3}{6}\\
u_4 &= \int_0^t (Du_3'' + A_3)dt = Da^3(1-\frac{2}{N})\{(1-\frac{2}{N})(6x-\frac{12}{N}x^2) + 2(1-\frac{x}{N})^2\\
&+ 6(1-\frac{1}{N})(1-\frac{x}{N})x + 2x^2(1-\frac{1}{N})^2(1-\frac{x}{N}) + D(1-\frac{2}{N})^2x^2 + \frac{a^2}{D}(1-\frac{2}{N})^2(1-\frac{x}{N})x^4\\
&+ \frac{a}{D}(1-\frac{2}{N})(1-\frac{x}{N})^2x^3\}\frac{t^4}{24} + Da^3(1-\frac{2}{N})^2[D(1-\frac{x}{N})^2x + a(1-\frac{x}{N})x^3]\frac{t^4}{8}
\end{aligned}$$

and so on, the solution $u(t,x)$ in the series form is given by

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x).$$

Example 3. Consider the nonlinear parabolic problem (2-3) with initial condition $u(0,x) = \sin(x)$ thus we have

$$\begin{aligned}
u_0 &= \sin x\\
u_1 &= \int_0^t (Du_0'' + A_0)dt = (-D\sin x + a\sin x(1-\frac{\sin x}{N}))t\\
u_2 &= \int_0^t (Du_1'' + A_1)dt = [D(D\sin x - a\sin x - \frac{2a}{N}(\cos^2 x - \sin^2 x))\\
&+ a\sin x(1-\frac{2}{N})((a-D)\sin x - \frac{a}{N})\sin^2 x]\frac{t^2}{2}\\
u_3 &= \int_0^t (Du_2'' + A_2)dt = [D(D-D^2)\sin x + \frac{8aD}{N}(1-2\sin^2 x) + aD(1-\frac{2}{N})\sin^2 x\\
&+ 2\frac{a^2}{N}(1-\frac{2}{N})\sin x(1-2\sin^2 x) - aD(1-\frac{2}{N})\cos^2 x + a^2(1-\frac{2}{N})\cos^2 x\\
&- 2\frac{a^2}{N}(1-\frac{2}{N})\sin x\cos^2 x - a(a-D)(1-\frac{2}{N})\sin^2 x + \frac{a^2}{N}(1-\frac{2}{N})\sin^3 x\\
&+ a(a-D)(1-\frac{2}{N})\cos^2 x - \frac{a^2}{N}(1-\frac{2}{N})\sin x\cos^2 x + a(D^2-aD)(1-\frac{2}{N})\sin^2 x\\
&- 2\frac{a^2D}{N}(1-\frac{2}{N})\sin x(1-2\sin^2 x) - a^2(D-a)(1-\frac{2}{N})^2\sin^3 x + \frac{a^3}{N}(1-\frac{2}{N})^2\sin^4 x\\
&+ a(a-D)^2(1-\frac{2}{N})\sin^2 x + \frac{a^3}{N^2}(1-\frac{2}{N})\sin^4 x - 2\frac{a^2}{N}(1-\frac{2}{N})(a-D)\sin^3 x]\frac{t^3}{6}
\end{aligned}$$

and so on,the solution $u(t,x)$ is in the following form

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x).$$

4 Conclusion

In this paper, the Adomian decomposition method has been applied for solving reaction diffusion equation. For an unbounded domain in any dimension, we recover the power series formula of [4], which is much simpler than the classical formula. Further, we accommodate the decomposition method to deal with Dirichlet, Neumann, and mixed boundary conditions. In comparison with other tools this method, give us a very better approximate solution.

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