Decay of Correlations on Non-Hölder Observables

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Abstract: This paper is devoted to investigating decay of correlations for hyperbolic systems with singularities on general continuous observables. We study the dependence of decay of correlations on the regularity of observables by the coupling method.

Keywords: Decay of correlations, hyperbolic systems, coupling method

1 Introduction

Let $M$ be a 2-d open connected manifold, such that the closure of $\Omega$ is compact. Assume there exists a closed subset $S_0$ such that $M \setminus S_0$ is open and dense in $M$. The map $T : M \setminus T^{-1}S_0 \rightarrow M \setminus T S_0$ is $C^{1+\gamma}$ diffeomorphism on each connected component in $M \setminus T^{-1}S_0$, for some $\gamma > 0$, that preserves a probability measure $\mu$.

The most interesting invariant measures are the so-called physical measures, or the SRB measures, after Sinai, Ruelle and Bowen, who introduced this notion over 30 years ago. More precisely, $\mu$ is called an SRB measure if its conditional measure on any unstable manifold is absolutely continuous with respect to the Lebesgue measure on it and the basin $B(\mu)$ has positive Lebesgue measure. Here the basin of a measure $\mu$ is defined as

$$B(\mu) = \{x \in M | \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \mu(f), \forall f \in C(M)\}$$

where $\mu(f) = \int f \, d\mu$ and $C(M)$ is the set of all continuous functions on $M$. If a system preserves an SRB measure $\mu$, the definition of the basin provides a way of visualizing the SRB measures for physicists by averaging the atom measures along a typical trajectory. The existence and finitude of mixing SRB measures for systems with singularities were derived by Alves, Bonatti, Chernov, Dolgopyat, Pesin, Sataev, Sinai, Viana, Young and many other experts. Here we only list a few such as [1, 6, 16, 18, 19].

The mixing property implies that we do not have to take the average when starting from a properly chosen initial probability distribution $\nu$ since $T^n \nu$ itself converges to $\mu$ as $n$ goes to $\infty$. More precisely, for any bounded observables $f \in L_\infty(M, \mu)$,

$$\lim_{n \to \infty} T^n \nu(f) = \mu(f). \quad (1)$$

In fact the mixing speed is characterized by the rates of decay of correlations defined as below. For any pair of integrable functions (observables) $f, g$ in $L^2(M, \mu)$, the correlations of $f \circ T^n$ and $g$ are defined by

$$C_{f,g}(n) = |\int_M (f \circ T^n) g \, d\mu - \int_M f \, d\mu \int_M g \, d\mu|, \quad n \in \mathbb{N}. \quad (2)$$

It is well known that $T : M \to M$ is mixing if and only if

$$\lim_{n \to \infty} C_{f,g}(n) = 0, \quad \forall f, g \in L^2_\mu(M). \quad (3)$$

A natural question related to the convergence is to characterize the speed of convergence. Standard counterexamples show that in general there is no specific rate at which the correlations converge to zero occurs. A natural question is what
is the particular speed at which the correlation function $C_{f,g}(n)$ decays for observables $f, g$ with sufficient regularity. In recent years many researchers are devoted to understanding the correlations decay rates for hyperbolic systems. Most of the studies are restricted to Hölder continuous functions, see [1, 4, 6, 7, 9, 19, 20]. The rates of mixing in systems with singularities is rather difficult to establish, though, because the singularity aggravates the analysis and make standard approaches (based on Markov partitions and transfer operators) hard to apply. For planar Sinai billiards, L.-S. Young [19] developed in 1998 a novel method (now known as Young's tower construction) to prove exponential (fast) mixing rates. Young applied it to Sinai billiards with finite horizon. Her method was later extended to all planar Sinai billiards by Chernov [4]. The coupling method was used by Young in [20] and improved by Chernov and Dolgopyat in [6, 11] which is designed to directly control the dependence between the past and the future geometrically. Both methods were proven to be very efficient and produced many sharp results. The tower method was applied to a number of systems even including suspension flows by Chernov, Demers, Liverani, Markarian, Melbourne, Török and many other experts, see for example [8, 10, 14, 15, 19, 20].

Here we are devoted to depict the following scenario: the explicit relation between the rate of correlations and the regularity of observables. In this work we continue to investigate system $(T, M)$, which was studied in [9] and we are able to study the decay of correlations by the coupling methods and obtain decay rates on general bounded observables including non-Hölder functions. In fact we obtain an estimation on precisely how the regularity of the observables affects the decay rates, see (14) in Theorem 1. Hopefully, this will lead us to have a better understanding on the relation between chaotic properties and statistical properties.

In this paper we mainly use the coupling method not only for convenience. The advantage for coupling method is that it enables us to have an efficient estimations on the speed of decay of correlations for different classes of observables. For a general observable $g$, we first take its average on each unstable leaves and thus define a new observable $\bar{g}$ which is constant on each leaf. By the coupling method, we are able to study the mixing speed of the probability measure with density functions obtained from normalizing $\bar{g}$ on each unstable leaf. Since the distribution we start with is uniform on each unstable manifold, one gets the exponential coupling rate. Finally one adds up the effect on the mixing rates caused by the difference $g - \bar{g}$, which is essentially due to the regularity of the function $g$. Furthermore, the results were applied to dispersing billiards and Arnold map.

The plan of this paper is the following. After listing sufficient conditions (H1-H4) for the system (see Section 2), we state the main results on estimations of decay rates for three types of functions. In Section 3 the concepts of standard pairs and families are introduced. In addition we specify a set of functions $\mathcal{D}$ which is invariant under the transfer operator. The coupling lemma is taken from paper [9] and stated in Section 4. The main theorem is proved in Section 5. Lastly, as an application, we apply the main theorem on the Arnold map, and shows that the upper bounds of the the correlations can be slow on non-Hölder observables.

Of course it is ideal to also obtain the lower bound for the decay rates of correlations. But to achieve the goal, substantial work is needed, and we will leave it at a later study.

NOTATION: Throughout the paper we will use the following conventions: Positive and finite global constants whose value is unimportant, will be denoted by $c, c_1, c_2, ...$ or $C, C_1, C_2, ...$. These letters may denote different values in different equations throughout the paper.

2 Statement of results

2.1 List of assumptions.

Let $d(\cdot, \cdot)$ be the distance function on $M \times M$ induced by the Riemannian metric in $M$ and $m$ be the corresponding Lebesgue measure on $M$. For smooth curve $W \subset M$, denote $m_W$ as the Lebesgue measure induced by the Riemannian metric on $W$. In particular, denote $|W| = m_W(W)$ as the Lebesgue length of $W$. We list the assumptions for our main theorems.

(H.1) Hyperbolicity of $T$. There exist two families of continuous cones $C^u_x$ (unstable) and $C^s_x$ (stable) in the tangent spaces $T_x M$ with the following properties:

(i) The angle between $C^u_x$ and $C^s_x$ is uniformly bounded away from zero for any $x \in M$;
(ii) $DT(C^u_x) \subset C^u_T$ and $DT(C^s_x) \supset C^s_T$ whenever $DT$ exists.

(iii) There exists $\Lambda > 1$ such that

$$
\|D_xT(v)\| \geq \Lambda \|v\|, \forall v \in C^u_x \quad \text{and} \quad \|D_xT^{-1}(v)\| \geq \Lambda \|v\|, \forall v \in C^s_x
$$

(4)

(H.2) Singularities.

Let $S_{\pm 1} = S_0 \cup T^{\pm 1}S_0$ be the singular set of $T^{\pm 1}$ and make the following assumptions:

(i) $S_1$ consists of finitely or countably many $C^2$ smooth curves.

(ii) The singular curves in $S_0$ can only terminate on each other, or on the boundary of $M$.

(iii) The angle between tangent vectors of $S_0$ and $C^u$ has a positive lower bound.

(iv) there exist $c > 0$, $p \in (0, 1]$ such that for any $\varepsilon > 0$

$$
m(x \in M : d(x, S_0) \leq \varepsilon) \leq c\varepsilon^p
$$

(5)

Remark: It follows from the Borel-Cantelli Lemma and (iv) that there are plenty of long stable manifolds and unstable manifolds in $M$:

$$
\mu(|W^{s,u}(x)| \leq \varepsilon) \leq C\varepsilon^p, \quad \forall \varepsilon > 0
$$

(6)

where $\mu$ is any $T$-invariant measure. We say that a smooth curve $W$ is an unstable or $u$-curve if at every point $x \in W$ the tangent line $T_xW$ belongs to the unstable cone $C^u_x$. If $T^{-n}(W)$ is an unstable manifold for all $n \geq 0$, then $W$ is an unstable manifold and denoted by $W^u$. Stable (or $s$-) curves and stable manifolds $W^s$ are defined in a similar way. In application to billiards, the singular set $S_0$ also contains some artificial singular curves including the boundary of Homogeneity stripes, see [7], to guarantee the distortion bounds. For convenience we assume that the lengths of unstable/stable curves are uniformly bounded by a small constant,

$$
|W| \leq c_M.
$$

(7)

To guarantee this we need also add finitely many fake singular lines in $M$.

Let $S_{\pm n} = \bigcup_{m=0}^n T^{\pm n}S_0$ and $S_{\pm \infty} = \bigcup_{m \geq 0} S_{\pm m}$. We denote $W^s$ as the collection of all stable manifolds in $M \setminus S_{-\infty}$ and $W^u$ as the collection of all unstable manifolds in $M \setminus S_{\infty}$. Note that any curve $W \in W^s/W^u$ has no end points, since both stable and unstable manifolds are terminated on $S_{\pm \infty}$. We assume that both stable and unstable manifolds form angles $\geq \alpha_0 > 0$ with singular curves in $S_{\pm 1}$ at intersection points.

(H.3) Regularity of smooth $s/u$-curves.

We assume that there is a class of $T$-invariant $s/u$-curves $W \subset M$ that are regular in the following sense:

(i) Bounded curvature. There exists a small constant $k_0 > 0$ such that at any point $x \in W$ the curvature of $W$ at $x$ is bounded from above by $k_0$.

(ii) Distortion bounds of $T$. There exist $C_1 > 1$, $\gamma \in (0, 1)$, such that for any unstable curve $W \subset M$, with $T^{-n}W$ also being an unstable curve, for some $n \geq 1$, and any $x, y \in W$,

$$
|\ln J_W T^{-1}(x) - \ln J_W T^{-1}(y)| \leq C_1 d_W(x, y)\gamma
$$

(8)

where $J_W T^{-1}$ is the Jacobian of $T^{-1}$ at $x$ along $W$.

(iii) Absolute continuity. For any two close enough regular unstable curve $W^s$ and $W^u$, denote

$$
W^s = \{x \in W^s : W^s(x) \cap W^{3-s} \neq \emptyset\}
$$

for $i = 1, 2$. The unstable holonomy map $h : W^s \rightarrow W^u$ along stable manifolds is absolutely continuous and has uniformly bounded Jacobian $J_{W^s} h$. Furthermore, for any $x, y \in W^s$,

$$
|\ln J_{W^s} h(x) - \ln J_{W^s} h(y)| \leq C_2 d_W(x, y)\gamma.
$$

(9)
Note that (8) and the uniform hyperbolicity implies that for any unstable manifold $W \subset M$, and any $k \geq 1$,
\[
|\ln J_W T^{-1}(T^{-k}x) - \ln J_W T^{-1}(T^{-k}y)| \leq C_d d_W(T^{-k}x, T^{-k}y) \leq c_M C_1 \Lambda^{-k\gamma}.
\]

(10)

Accordingly, for any $n \in \mathbb{N}$ and any unstable curve $W$ such that $T^n W$ is smooth, the expansion factor is almost constant on $W$. In applications, in order to make sure that a given unstable curve $W$ to be regular, one needs to add finitely many grids in $M$ and thus to make sure $c_M$ is small. From now on, all the $s/u$-curves we mention refer to the regular $s/u$-curves without emphasizing.

(H.4) One-step expansion. Assume
\[
\lim \inf_{\delta_0 \to 0} \sup_{W: |W| < \delta_0} \sum_{\beta \in W/\xi^1} \left( \frac{|W|}{|V_{\beta}|} \right)^p \frac{|T^{-1}V_{\beta}|}{|W|} < 1,
\]
where the supremum is taken over all unstable manifolds $W \in \mathcal{W}^u$, $\xi^1$ is the partition of $M$ into connected components in $M \setminus S_1$, and $\{V_{\beta}: \beta \in W/\xi^1\}$ is the collection of all smooth components in $TW \setminus S_{-1}$. Now the existence of finitely many mixing, Bernoulli SRB measures follows from the results from Pesin and Sateav in [16, 18] for general hyperbolic $M$.

2.2 Main results

Let $Q(x)$ be the open connected component of the set $M \setminus S_0$ containing the phase point $x$. For any $x, y \in M$ denote the forward (resp. backward) separation time of $x$ and $y$ by
\[
s_+(x, y) = \min\{n \geq 0: T^n y \notin Q(T^n x)\} \quad \text{and} \quad s_-(x, y) = \min\{n \geq 0: T^{-n} y \notin Q(T^{-n} x)\}
\]

(12)

Observe that if $x$ and $y$ lie on one unstable curve $W$, then by the uniform expansion property and the distortion bounds
\[
d_W(x, y) \leq C \Lambda^{-s_+(x, y)}
\]

(13)

where $d_W(x, y)$ is the distance between $x$ and $y$ in $W$ and $C = C(M) > 0$.

Let $\Psi$ be the space of all continuous functions $\psi: (0, \infty) \to (0, \infty)$ with the following properties:

1. $\lim_{t \to 0^+} \psi(t) = 0$;
2. There exists $\hat{c} > 0$ such that $\psi(d_W(x, y)) \leq \hat{c} \psi(\Lambda^{-s_+(x, y)})$.

For any $\psi \in \Psi$, let $\mathcal{H}^+(\psi)$ be the set of all bounded Borel functions $f: M \to \mathbb{R}$ such that for any unstable manifold $W$,
\[
|f(x) - f(y)| \leq K_f \psi(d_W(x, y)), \quad \forall x, y \in W
\]

where $K_f > 0$ is a constant depends on $f$. For any $f \in \mathcal{H}^+(\psi)$, let
\[
\|f\|_{\psi,+} = \sup_{W \in \mathcal{W}^u} \sup_{x, y \in W} \frac{|f(x) - f(y)|}{\psi(d_W(x, y))}
\]

where $\mathcal{W}^u$ is the collection of all smooth unstable manifolds. Similarly, we define $\mathcal{H}^-(\psi)$ as the set of all bounded Borel functions $f: M \to \mathbb{R}$ such that for any stable manifold $W$,
\[
|f(x) - f(y)| \leq K_f \psi(d_W(x, y)), \quad \forall x, y \in W
\]

where $\mathcal{W}^s$ is the collection of all smooth stable manifolds. Let $\mathcal{H}^\pm = \cup_{\psi \in \Psi} \mathcal{H}^\pm(\psi)$. Let $\|\cdot\|_\pm$ be a norm on $\mathcal{H}^\pm(\psi)$ such that for any $f \in \mathcal{H}^\pm(\psi)$,
\[
\|f\|_\pm = \|f\|_\infty + \|f\|_{\psi,\pm}
\]

One can see that the regularity of functions $f \in \mathcal{H}^\pm(\psi)$ is characterized by the function $\psi \in \Psi$, in particular by the speed of convergence of $\psi(\Lambda^{-n})$ to zero as $n \to \infty$. 
Theorem 1 Assume the conditions (H.1-H.4) hold. If $(T^n, \mu)$ is ergodic for any $n \geq 1$ then there exists $s \in (0, p]$ such that for any $f \in \mathcal{H}^-(\psi)$ and any $g \in \mathcal{H}^+(\psi)$ with $\psi_1, \psi_2 \in \Psi$,
\[
\mathcal{C}_{f,g}(n) \leq C_1\|g\|_\infty \|f\|_{\Psi_1} \psi_1(n) + C_2\|g\|_\infty \|f\|_{\Psi_1} - \psi_1(n) + C_3\|f\|_\infty \|g\|_{\Psi_2} \psi_2(n) + \psi_2(n)
\]
where $\vartheta = \Lambda^{-1/2}$.

Next we derive a corollary from the above theorem.

Corollary 2 Assume $(T, \mu)$ is mixing and $f \in \mathcal{H}^-(\psi)$ and $g \in \mathcal{H}^+(\psi)$ for some $\psi \in \Psi$.

(a) If $\psi(t) \leq ct^s$ then
\[
\mathcal{C}_{f,g}(n) \leq C\|f\|_\infty \|g\|_\infty \psi(n)
\]
(b) If $\psi(t) > ct^p$ for any $c > 0$, then the decay rates are dominated by $\psi(\vartheta^n)$:
\[
\mathcal{C}_{f,g}(n) \leq C\|f\|_\infty \|g\|_\infty \psi(\vartheta^n)
\]

In particular, the decay rates can be arbitrarily slow:

(1) If $\psi(t) = O(e^{\ln t})$ for some $\sigma \in (0, 1)$, then
\[
\mathcal{C}_{f,g}(n) \leq C\|f\|_\infty \|g\|_\infty \Lambda^{-\sigma}
\]

(2) If $\psi(t) = O(|\ln t|^\sigma)$ for some $\sigma > 0$, then
\[
\mathcal{C}_{f,g}(n) \leq C\|f\|_\infty \|g\|_\infty \Lambda^{-\sigma}
\]

As a remark here, for smooth systems, the assumption (H.4) is not needed to get the main theorem. The main reason that we assume (H.4) is to grantee the uniform growth (on average) in every step. For smooth hyperbolic map, such as Anosov diffeomorphism, the uniform hyperbolicity and ((H.1)-(H.3)) are enough to obtain Theorem 1.

3 Standard families

For a fixed large constant $C_r > C_1$, denote $\mathcal{D}$ as the set of all functions $f : M \to (0, \infty)$ such that there exists a measurable foliation $W$ of $M$ into unstable curves, and for any $W \in \mathcal{W}$ and $x, y \in W$,
\[
|\ln f(x) - \ln f(y)| \leq C_r d_W(x, y)\gamma.
\]

The main reason that we define the space $\mathcal{D}$ is that all of our probability measures will have density functions belong to $\mathcal{D}$, including all images of probability measures conditioned from the Lebesgue measure. Furthermore let $W$ be any u-manifold, we say that $(W, \nu)$ is a standard pair if $\nu$ a probability measure supported on $W$ and is absolutely continuous with respect to the Lebesgue measure $m_W$, such that the density function $d\nu/dm_W \in \mathcal{D}$.

Let $\mathcal{L}$ be the Ruelle-Perron-Frobenius transfer operator acting on the space of functions $\mathcal{D}$, such that for any $g \in \mathcal{D}$ equipped with a measurable partition $\mathcal{W}$ of $M$:
\[
(\mathcal{L}g)(y) = g(T^{-1}y)J_W T^{-1}(x), \quad \forall y \in T\mathcal{W}
\]

Next we will show that $\mathcal{D}$ is essentially $\mathcal{L}$-invariant.

It can be shown that there exists $N_0 = 1$ such that for any $n \geq N_0$, we have $\mathcal{L}^n \mathcal{D} \subset \mathcal{D}$. For simplicity we always assume $N_0 = 1$. One of the main reason that we consider standard pair $(W, \nu)$ is that it is associated with any probability measure $\nu$ that is indeed equivalent to $\nu_W$, which is induced by the Lebesgue measure $m_W$. More precisely, if $(W, \nu)$ is a standard pair with $T^nW$ smooth, for some $n \in \mathbb{N}$, then:
\[
e^{-C_r\Lambda^{-\gamma}} \nu_W(A) \leq \nu(A) \leq e^{C_r\Lambda^{-\gamma}} \nu_W(A)
\]

Clearly, as $n$ gets larger, the measure $\nu$ becomes almost uniform on $W$. In particular we define
\[
c_r = e^{C_r}
\]

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Note that for any standard pair \((W, \nu)\), \(T\nu\) is a measure supported on \(TW = \cup_{\alpha \in \mathcal{A}} V_\alpha\). We extend the transformation \(T\) on \((W, \nu)\) as follows: the image of \((W, \nu)\) under \(T\) can be viewed as a collection of pairs \(\{(V_\alpha, \nu_\alpha) : \alpha \in \mathcal{A}\}\), weighted by a factor measure \(\lambda\) on the index set \(\mathcal{A}\). Now we extend our definition of standard pairs to standard families.

Let \(\{(W_\alpha, \nu_\alpha) : \alpha \in \mathcal{A}\}\) be a (countable or uncountable) family of standard pairs with \(W = \{W_\alpha | \alpha \in \mathcal{A}\}\). We call it a standard family if there exists a probability factor measure \(\lambda\) on \(\mathcal{A}\), which defines a measure \(\nu_\mathcal{G}\) supported on \(\mathcal{C}(W) := \{x \in W\}\) by

\[
\nu_\mathcal{G}(B) = \int_{\alpha \in \mathcal{A}} \nu_\alpha(B \cap W_\alpha) \, d\lambda(\alpha) \tag{17}
\]

For all measurable set \(B \subset \Omega\). For simplicity, we denote a standard family by \(\mathcal{G} = (W, \nu_\mathcal{G})\). Fix a large constant \(C_p > C_r\), we say that \(\mathcal{G}\) is a proper (standard) family, if

\[
\int_{\alpha \in \mathcal{A}} |W_\alpha|^{-p} \, d\lambda(\alpha) < C_p.
\]

An intuitive way to look at the standard family is to view it as a decomposition of the measure \(\nu_\mathcal{G}\) along a measurable partition \(\mathcal{W}\) of the support of \(\nu_\mathcal{G}\). In fact we can show that any density function \(g \in \mathcal{D}\) induces infinitely many standard families.

**Lemma 3** Let \(g \in \mathcal{D}\) with support \(B \subset M\) and \(d\nu_\alpha = g \, d\nu_\|\) be the probability measure associated with \(g\). For any measurable partition \(\mathcal{W}\) of \(B\) into \(u\)-manifolds, there exists a standard family \((\mathcal{W}, \nu_g)\) equipped with a factor measure \(\lambda\) on the index set of \(\mathcal{W}\).

Proof: Let \(g \in \mathcal{D}\), and denote \(\mathcal{W} = \{W_\alpha : \alpha \in \mathcal{A}\}\) to be any measurable partition of the support of \(g\) into \(u\)-manifolds. Then there exists a factor measure \(\lambda\) (depends on the partition \(\mathcal{W}\) and \(g\)) defined on the index set \(\mathcal{A}\) equipped with the Borel \(\sigma\)-algebra induced from \(M\), such that for any measurable set \(A_1 \subset \mathcal{A}\),

\[
\lambda(A_1) = \nu_g(x \in W_\alpha : \alpha \in A_1)
\]

Accordingly, the measure \(\nu_g\) has a decomposition:

\[
\nu_g(A) = \int_{\alpha \in \mathcal{A}} \int_{x \in W_\alpha \cap A} g_\alpha(x) \, dm_{W_\alpha}(x) \, d\lambda(\alpha), \tag{18}
\]

for any Borel set \(A \subset M\). Here \(g_\alpha\) is the density function of a probability measure \(\nu_\alpha\) on \(W_\alpha\), i.e., for any \(x \in W_\alpha\),

\[
g_\alpha(x) := g(x) / \int_{W_\alpha} g(x) \, dm_{W_\alpha}.
\]

The fact that \(g \in \mathcal{D}\) implies that \(g_\alpha \in \mathcal{D}\). Thus \((W_\alpha, \nu_\alpha)\) is a standard pair. By definition \(\mathcal{G} = (\mathcal{W}, \nu_\mathcal{G})\) is indeed a standard family. edq

The distortion bound implies that if \(\mathcal{G} = (\mathcal{W}, \nu_\mathcal{G})\) is a standard family with a factor measure \(\lambda\) on the index set \(\mathcal{A}\), then \(T^n\nu_\mathcal{G}\) also induces a standard family with \(T^n\mathcal{G} = (T^n\mathcal{W}, T^n\nu_\mathcal{G})\), where for \(n \geq 0\) and measurable set \(B \subset \Omega\),

\[
T^n\nu_\mathcal{G}(B \cap T^n\mathcal{W}) := \int_{\alpha \in \mathcal{A}} T^n\nu_\alpha(B \cap T^n\mathcal{W}) \, d\lambda(\alpha). \tag{19}
\]

Similarly, for any \(n \geq 1\), \(\mathcal{L}^n g\) also generates a probability measure \(\nu_{\mathcal{L}^n g}\) associated with the standard family \((T^n\mathcal{W}, \nu_{\mathcal{L}^n g})\). Now for any measurable set \(B \subset M\),

\[
T^n\nu_g(B \cap T^n\mathcal{W}) = \int_{\alpha \in \mathcal{A}} T^n\nu_\alpha(g \cdot \chi_B) \, d\lambda(\alpha)
= \int_{\alpha \in \mathcal{A}} \nu_\alpha(\mathcal{L}^n g \cdot \chi_B) \, d\lambda(\alpha) = \nu_{\mathcal{L}^n g}(B \cap T^n\mathcal{W}) \tag{20}
\]

### 4 Coupling lemma

Given any standard family \((\mathcal{W}, \nu)\), one intuitive question to know is the average length of curves in \(\mathcal{W}\), or the distribution of short unstable manifolds in \(\mathcal{W}\) corresponding to \(\nu\). One of the fundamental facts in the theory of chaotic billiards is the Growth Lemma that describes the victory of the hyperbolicity over singularities: expansion always prevails fragmentation.

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We now turn to the precise definition. Let $W$ be any unstable curve in $M \setminus S_t$, for any $x \in W$ denote $W_n(x) \subset T^nW$ as the regular unstable curve that contains $x_n := T^n x$ and define
$$r_{W,n}(x) = d_{W_n(x)}(x_n, \partial W_n(x))$$

The following lemma was proved in [9] under condition (H1-H4). In fact in lemmas proved in this paper, the key assumption that guarantees the three properties (1)-(3) is the one-step expansion condition (11).

**Lemma 4** (Growth Lemma) The system $(T, M, \mu)$ has the following properties:

1. There exist $c > 0, C_\varepsilon > 0$ and $\theta \in (0, 1)$, such that for any standard pair $\mathcal{G} = (W, \nu)$ and $\delta > 0$,
   $$m_W(r_{W,n}(x) < \delta) \leq (e^{\delta^n} + C_\varepsilon |W|)\delta^p$$

2. There exists $\chi > 0$ such that for any standard pair $\mathcal{G} = (W, \nu)$ and $n > \chi \ln |W|$, $T^n \mathcal{G}$ is proper.

3. There exist $c_1 > 0$ and $\theta \in (0, 1)$, such that for any proper family $\mathcal{G} = (W, \nu)$ and $\delta > 0$,
   $$\nu(r_{W,n}(x) < \delta) \leq c_1 \delta^p$$

4. The family $\mathcal{G} = (W, \mu)$ is proper.

Another important fact follows from the one-step-expansion condition (11) proved in [6, 7, 9] on the existence of plenty of long stable manifolds for points belong to a proper family. For any $x \in M$, let $r^\sigma(x) = d_{W^\sigma(x)}(x, \partial W^\sigma(x))$, where $W^\sigma(x)$ is the stable (resp. unstable) manifold that contains $x$, for $\sigma \in \{s, u\}$.

**Lemma 5** There exists $C > 0$ such that for any $\delta > 0$, any proper family $\mathcal{G} = (W, \nu)$ of $T$,
$$\nu(r^\sigma(x) < \delta) \leq C \delta^p$$

In particular this implies a similar argument as (6) for $(T, \mu)$, if we take $\mathcal{W}$ to be the partition of $M$ into unstable manifolds of $T$, then apply the above lemma for the proper family $(W, \mu)$. According to [6, 7, 9], under assumptions (H1-H4), there exists a hyperbolic set $R^s \subset M$ with a positive Lebesgue measure.

Let $(W, \nu)$ be a standard pair, and define $\tilde{W} = W \times [0, 1]$, which is a rectangle based on $W$. We equip $\tilde{W}$ with a probability measure $\tilde{\nu}$, such that for any $(x, t) \in \tilde{W}$,
$$d\tilde{\nu}(x, t) := d\nu(x)dt.$$ (23)

Note that the map $T^n$ defined on $W$ can be extended to $\tilde{W}$ with $T^n(x, t) := (T^n(x), t)$. Let $\tilde{\mathcal{G}} = (\tilde{W}, \tilde{\nu})$ be a standard family we define $\tilde{W} = W \times [0, 1]$ and $\tilde{\mathcal{G}} = (\tilde{W}, \tilde{\nu})$, where the probability measure $\tilde{\nu}_\mathcal{G}$ satisfies $d\tilde{\nu}_\mathcal{G}(x, t) = d\nu(x)dt$, for all $(x, t) \in \tilde{W}$. Let $\pi_1 : \tilde{W} \to W$ be the natural projection such that for any $(x, t) \in \tilde{W}$, $\pi_1(x, t) = x \in W$.

**Lemma 6** (Coupling lemma) Let $\nu^i \in \mathcal{D}, i = 1, 2$, be proper measures that generate $(W^i, \nu^i)$. Under assumption (H1-H4), there exist a decomposition of $W_i = \sum_{n=1}^{\infty} W_n^i$ and a bijection $\Theta : W_1 \to W_2$ with the following properties:

1. $\Theta W_n^1 = W_n^2$ and $\Theta \nu_n^1|_{W_n^1} = \nu^2|_{W_n^2}$ for all $n \in \mathbb{N}$;

2. $T^n(\pi_1 W_n^1)$ is an $\varepsilon$-subset of $\Gamma^*$;

3. If $\Theta(x_1, t_1) = (x_2, t_2)$, for $(x_1, t_1) \in W_n^1$, then $T^nx_2 \in W^*(T^n x_1)$.

Furthermore (H4) implies that there exist $C_\varepsilon > 0$ and $s \in (0, p]$ such that for any $n \geq 1$,
$$\hat{\nu}^1(W_n^1) < C_\varepsilon A^{-sn}, \quad i = 1, 2.$$ (24)

## 5 Decay of correlations.

In this section we study the rate of decay of correlations for the system $(T, M, \mu)$ by the coupling methods, which leads to an estimation of the correlation rates on a much larger class of observables.

The next proposition describes the equidistribution property of the SRB measure. Namely, for any probability measure $\nu$ associated with a proper family $\mathcal{G}$, the image $T^n \nu$ weakly converges to the SRB measure $\mu$ on a large set of observables at a relatively fast rate. As explained in [6] that the equidistribution property effectively pronounces asymptotic independence between the present and the future of the system. Notice functions in $\mathcal{H}^\pm(\psi)$ are much general than backward/forward dynamically Hölder continuous. As a result, we might not get exponential decay rates on these general observables.
**Proposition 7 (Equidistribution)** Let \( (W^1, \nu^1) \) and \( (W^2, \nu^2) \) be two proper families. Then for any \( f \in H^-(\psi) \),

\[
| \int_{W^1} f \circ T^n \, d\nu^1 - \int_{W^2} f \circ T^n \, d\nu^2 | \leq C_1 ||f||_{\Lambda} - s^n + C_2 ||f||_{\psi} \Lambda^{-s^n},
\]

where \( C_1 > 0, C_2 > 0 \) are uniform constants and \( s \in [0, p] \) is given in Lemma 6.

Proof We first define the rectangular family \( \hat{W}^1 = (\hat{W}^1, \hat{\nu}^1) \) as (23). By the Coupling Lemma 6, there exists a decomposition \( \hat{W}^1 = \sum_{n=1}^{\infty} \hat{W}^1_n \) such that \( \Theta : \hat{W}^1_n \to \hat{W}^2_n \) is measure preserving and \( T^n \pi_1 \hat{W}^1_n \) is a collection of \( u \)-subsets of \( \Gamma^u \). Furthermore if \( \Theta(x, t_1) = (y, t_2) \) then \( x \) and \( y \) belong to the same stable manifold in \( \Gamma^u \).

Accordingly for any \( f \in H^-(\psi) \) and \( n \geq m \)

\[
| \int_{\hat{W}^1_m} f \circ T^n(x, t_1) \, d\hat{\nu}^1(x, t_1) - \int_{\hat{W}^2_m} f \circ T^n(y, t_2) \, d\hat{\nu}^2(y, t_2) | \leq \hat{c} ||f||_{\psi} \Lambda^{-s^n} \hat{W}^1_m \leq \hat{c} C_1 ||f||_{\psi} \Lambda^{-s^n} \Lambda^{-s^n}
\]

Thus for any \( n \in \mathbb{N} \),

\[
| \int_{\hat{W}^1_m} f \circ T^n(x, t_1) \, d\hat{\nu}^1(x, t_1) - \int_{\hat{W}^2_m} f \circ T^n(y, t_2) \, d\hat{\nu}^2(y, t_2) | \leq \hat{c} C_1 ||f||_{\psi} \left( \sum_{m=1}^{n} \psi(\Lambda^{-s^n}) \Lambda^{-s^n} + \sum_{m=n+1}^{\infty} \right) \int_{\hat{W}^1_m} f \circ T^n \, d\hat{\nu}^2 - \int_{\hat{W}^1_m} f \circ T^n \, d\hat{\nu}^1 | \ (25)
\]

By the definition of the class of functions \( \Psi, \psi \) has bounded directives. It follows from summation by parts, there exists \( C_1 > 0 \) such that

\[
\sum_{m=1}^{n} \psi(\Lambda^{-s^n}) \Lambda^{-s^n} \leq \Lambda^{-s^n} \psi(1) + \psi(\Lambda^{-n}) + B \sum_{m=1}^{n} \Lambda^{-s^n} \Lambda^{-s^n} \leq \Lambda^{-s^n} \psi(1) + \psi(\Lambda^{-n}) + B C_1 \Lambda^{-s^n} = \psi(\Lambda^{-n}) + (\psi(1) + BC_1) \Lambda^{-s^n}
\]

This finishes the proof of Proposition 1.

It follows from this Proposition, the convergence rates of \( T^n \nu(f) \) to \( \mu(f) \) depend heavily on the regularity of the observable \( f \). Next we extend our study on the rate of decay of correlations on class of observables \( g \in D \) that are not proper. Note that for any \( n \geq 0 \) and \( g \in D \), \( L^n g \) also induces a standard family \( (T^n W^m, \nu_{C^n}) \). Here we denote \( d\nu_g = gd\nu \). Now we are ready to prove Theorem 1.

Let \( f \in H^+(\psi_1) \) and \( g \in H^+(\psi_2) \). Denote \( W = W^m \) as the measurable partition of \( M \) into smooth unstable manifolds. Fix a large \( n \) and denote

\[
W' = \{ W \in W | |W| \geq \Lambda^{-n/2} \}
\]

with \( A_n \) as the index set of \( W' \) and \( A_n = \{ x \in W' \} \). Assume \( g \geq 0 \) and \( \int_{A_n} g(x) \, d\mu(x) \neq 0 \). Then

\[
g_1 := g/\int_{A_n} g(x) \, d\mu(x)
\]

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generates an essentially proper family $\mathcal{G} = (\mathcal{W}', \nu_{\beta_1})$. In particular by Lemma 4, $T_n^{n/2}\mathcal{G}$ is proper. Now we can apply Proposition 7 to obtain the corresponding decay rates.

Note that for any $W_\beta \in \mathcal{W}'$, let $\{V_{\beta,i}\}$ be the collection of smooth components in $W_\beta \setminus \mathcal{S}_{n/2}$, then $T_n^{n/2}V_{\beta,i}$ is also a smooth component in $T_n^{n/2}W_\beta$. We define a new function which is constant on each $W_\beta \in \mathcal{W}'$:

$$\tilde{g}(x) = \int_{V_{\beta,i}} g_1(x) \, dm_{W_\beta}, \quad \forall x \in V_{\beta,i} \subset W_\beta, \, i \geq 1$$

Then $\tilde{g}$ is constant on each smooth component of $W_\beta \setminus \mathcal{S}_{n/2}$. Thus one can show that

$$\sup_{\beta \in \mathcal{A}, x \in W_\beta} |g_1(x) - \tilde{g}(x)| \leq \|g\|_{\psi_2} + \psi_2(\Lambda^{-n/2})$$

We also denote $g_2 = \tilde{g} \int_{A_\beta} g \, dm$. By Proposition 7 for $f \in \mathcal{H}^-(\psi_1)$ we have

$$|\int_{A_n} f \circ T^n \cdot g \, dm - \int_{A_n} f \, dm \int_{A_n} g \, dm|$$

$$\leq C' \|g\|_{\psi_2} + \|f\|_{\psi_2} \psi_2(\Lambda^{-n/2}) + \|g\|_\infty \int_{T^n A_n} (f \circ T^n) \cdot \tilde{g} \circ T^{-n/2} \, dm - \int_{A_n} f \, dm$$

$$\leq C' \|g\|_{\psi_2} + \|f\|_{\psi_2} \psi_2(\Lambda^{-n/2}) + C\|g\|_{\psi_2} + \|f\|_{\psi_2} \psi_2(\Lambda^{-n/2})$$

Combining the above facts we have

$$|\int_M f \circ T^n \cdot g \, dm - \int_M f \, dm \int_M g \, dm|$$

$$\leq C_1 \|g\|_\infty \|f\|_{\psi_1} \psi_1(\Lambda^{-n/2}) + C_2 \|g\|_{\psi_2} + \|f\|_{\psi_2} \psi_2(\Lambda^{-n/2}) + C\|g\|_{\psi_2} + \|f\|_{\psi_2} \psi_2(\Lambda^{-n/2})$$

For a general function $g = g_+ - g_-$ with $g_+ \geq 0$, we apply the above analysis to $g_-$ and $g_+$ separately and still get (14). This finishes the proof of Theorem 1.

6 Applications

As a by-product, corresponding to the case when the initial singular set is empty, we add artificial singular curves according to guarantee that we still get a family of regular stable/unstable manifolds. On the other hand, the above results are new even for the smooth case. Since even in the strongest chaotic systems one may expect to get arbitrarily slow decay of correlations on certain continuous observables.

Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the Arnold cat map on a torus $\mathbb{T}^2 = (0, 1) \times (0, 1)$. We divide $M = [0, 1] \times [0, 1]$ by finite number of horizontal and vertical lines, which are added into the set $\mathcal{S}_0$, such that any unstable manifold in $M \setminus \mathcal{S}_0$ has length less than $c_M$. It can be easily shown that for each $n \geq 1$, $\mathcal{S}_{n/2}$ contains only finitely many smooth curves. It can be shown that $T : M \to M$ is equivalent to the original Arnold map. It can be easily checked that (H.1)-(H.3) are automatically true, except (H.3). But note that assumption (H.3) is only used to guarantee the Growth Lemma, which is automatically true for the cat map. Since $T$ preserves the area $m$, so $\rho(x) \equiv 1$ is the density function of the SRB measure on $\mathbb{T}^2$.

Let $c_h$ be any horizontal curve in $\mathcal{S}_0$ and $c_v$ be any vertical curve, respectively. Let $W = T\mathcal{S}_0$ and $g(x) = \chi_W(x)$. Then $\tilde{g} = g \int_W g \, dm_W \chi_W$ belongs to $\mathcal{D}$ and is proper. Then $T^nW$ converges to $W^n$ in the Hausdorff metric, where $W^n$ is the partition of $M$ into unstable manifolds. Furthermore the convergence speed can be exponential or polynomially depending on the observables. i.e. there exists $s \in (0, 1]$ such that for any $\psi \in \Psi$, $n \geq 1$ and $f \in \mathcal{H}^-(\psi)$,

$$|T^n \nu_\beta(f) - m(f)| \leq C_1 \|f\|_{\infty} \Lambda^{-sn} + C_2 \|f\|_{-\psi} \Lambda^{-n}$$

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In particular, $T^n S_0$ converges to unstable manifolds in $\mathcal{W}^u$. This fact that can be checked by elementary calculations. Since curves in $S_0$ are straight line segments. It can be easily calculated that the slope of $T^n c_h$ and $T^n c_v$ satisfies
\[
\text{slope} \left( T^n c_h \right) = \frac{b_{2n-1}}{b_{2n}} \quad \text{and} \quad \text{slope} \left( T^n c_v \right) = \frac{b_{2n}}{b_{2n+1}}
\]
where $\{b_n\}$ is the Fibonacci sequence defined by the recurrence relation $b_n = b_{n-1} + b_{n-2}$ with seed values $b_0 = 0$ and $b_1 = 1$. On the other hand, the slope of the global unstable manifold has the golden ratio $\lambda$ as its slope:
\[
\lambda = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]
Notice that
\[
\frac{b_1}{b_2} = 1, \quad \frac{b_2}{b_3} = 1 + \frac{1}{1 + 1}, \quad \frac{b_3}{b_4} = 1 + \frac{1}{1 + 1 + 1}, \quad \frac{b_4}{b_5} = 1 + \frac{1}{1 + 1 + 1 + 1}, \ldots
\]
Accordingly both $T^n c_h$ and $T^n c_v$ converge to $\mathcal{W}^u$.

Let $f \in \mathcal{H}^-(\psi), g \in \mathcal{H}^+(\psi)$ with $\psi(\Lambda^{-n}) = O(n^{-b})$ for $b > 0$. Then the Arnold map only has arbitrarily slow rates of correlations depending on $b$:
\[
C_{f,g}(n) \leq C_{f,g} n^{-b}
\]

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