Self-Similarities and Period-Adding in the Parameter-Space of a Nonlinear Resonant Coupling Process

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Abstract: In this paper we investigate the dynamics of a set of three autonomous first-order nonlinear ordinary differential equations. This set models a nonlinear resonant coupling process which considers the nonlinear decay of a linearly unstable high-frequency wave into a linearly damped low-frequency wave [Physica D, 236 (1982)]. We analytically determine the location of the equilibrium points as a function of the parameters, and decide on its stability in the parameter-space. A numerical investigation is realized on the two-dimensional parameter-space. By computing largest Lyapunov exponents, we show that this parameter-space presents periodic structures organized in period-adding bifurcation cascades, embedded in a large chaos region. We also show that these periodic structures are self-similar.

Keywords: Parameter-space; Lyapunov exponents; period-adding; self-similar

1 Introduction

Interest in studies focusing two-dimensional parameter-spaces of continuous-time dynamical systems modeled by differential equations has increased in many fields, in recent years. In 2005, Bonatto et al. [1] reported results concerning the parameter-space of a CO₂ laser, whose principal discovery is that periodicity islands embedded in a large chaos region emerge organized in a very regular network of self-similar structures called shrimps [2]. Studies involving the parameter-space of a parametrically excited oscillatory system composed of a Mathieu oscillator and a damped harmonic oscillator coupled by nonlinear terms were reported [3], where periodic windows of complex structure (shrimps) spread in a chaotic region were also detected. A particular two-dimensional parameter-space of the Rössler system was investigated by fixing the value of one parameter [4]. Basic elementary cells, called swallows by the authors, were also detected in this case. The parameter-space of a semiconductor laser with optical injection was studied [5], and was shown that chaotic regions are riddled with stable periodic solutions self-organized in period-adding bifurcation cascades which accumulate towards specific boundaries. Families of period-adding bifurcation cascades embedded in a chaotic region were also found in parameter-space of a damped-driven Duffing oscillator [6], in parameter-space of a Chua circuit where the nonlinearity of the Chua diode is described by a piecewise linear function [7] or by a cubic polynomial [8], and in parameter-space of an electronic circuit which contains two diodes as nonlinearity [9]. Features similar to that detected in references above-mentioned were also found in parameter-space of differential equations that model chemical reactions [10]. As far as we know, only very recently these shrimps were observed in a real experiment involving a Chua circuit [11].

In this paper the investigation is mainly concerned with the parameter-space of a set of three autonomous first-order nonlinear ordinary differential equations that model a nonlinear resonant coupling process, which considers the nonlinear decay of a linearly unstable high-frequency wave into a linearly damped low-frequency wave [12]. We locate the equilibrium points as a function of the parameters, and perform a detailed analytical investigation about its stability. The largest Lyapunov exponent is used to numerically delimit regions in parameter-space, characterized by the existence of equilibrium points, periodic points, or chaotic points. As is well-known, a negative largest Lyapunov exponent indicates a stable equilibrium point, a zero largest Lyapunov exponent indicates a stable periodic attractor, and a chaotic attractor has a positive largest Lyapunov exponent.

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The paper is organized as follows. In Sec. 2 is presented the mathematical model of this autonomous continuous-time three-dimensional dynamical system. Some analytical results concerning the dissipativity condition, equilibrium points, and stability of the equilibrium points, are obtained in Sec. 3. Detailed numerical simulations for this system, when the two parameters are varied, involving the largest Lyapunov exponent and bifurcation diagrams, are presented in Sec. 4. Finally, the paper is summarized in Sec. 5.

2 The Mathematical Model

In this section we first consider a rapid derivation of the mathematical model for the nonlinear decay of a linearly unstable high-frequency wave into a linearly damped low-frequency wave (see Ref. [12]) and references therein for details). Let the two waves characterized respectively by the complex amplitudes

\[ A_0 = |A_0|e^{i\phi_0} \quad \text{and} \quad A_1 = |A_1|e^{i\phi_1}, \]

by the real frequencies \( \omega_0, \omega_1 \), by the growth rates \( \gamma_0 > 0, \gamma_1 < 0 \), and nonlinearly coupled through a real coefficient V. Additionally, if the two waves satisfy the coupled equations

\[
\begin{align*}
    i \left( \frac{dA_0}{dt} - \gamma_0 A_0 \right) &= V A_1^2 e^{-i\Delta\omega t} \quad \text{and} \quad i \left( \frac{dA_1}{dt} - \gamma_1 A_1 \right) = V A_0 A_1^* e^{+i\Delta\omega t},
\end{align*}
\]

with \( \Delta\omega = \omega_0 - 2\omega_1 \), then, by making

\[
\begin{align*}
    x &= \frac{V}{\gamma_0} |A_0| \sin \theta, \
    y &= \frac{V}{\gamma_0} |A_0| \cos \theta, \
    \theta &= \phi_0 - 2\phi_1 + \Delta\omega t', \\
    z &= \frac{V^2}{\gamma_0} |A_1|^2, \
    t &= \gamma t',
\end{align*}
\]

the set of three autonomous dimensionless form first-order nonlinear ordinary differential equations given by

\[
\begin{align*}
    \dot{x} &= x + \alpha y - z + 2y^2, \\
    \dot{y} &= y - \alpha x - 2xy, \\
    \dot{z} &= -2\gamma z + 2xz,
\end{align*}
\]

where \( \gamma = -\gamma_1/\gamma_0, \alpha = \Delta\omega/\gamma_0 \), and \( z > 0 \), can be derived.

3 Analytical Results

The divergence of the vector field (1) is given by

\[
\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = 2 - 2\gamma,
\]

and from this result we conclude that the system (1) is dissipative for \( \gamma > 1 \) signifying, therefore, that the phase-space contracts volumes as the time \( t \) increases. As a consequence, all the bounded system trajectories finally converge to an attractor in a three-dimensional phase-space, when \( \gamma > 1 \).

Now we determine the location of the equilibrium points of the nonlinear system (1), and discuss the stability. By doing \( \dot{x} = 0, \dot{y} = 0, \) and \( \dot{z} = 0 \), we obtain two equilibrium points, namely

\[ P_0 \equiv (0, 0, 0) \quad \text{and} \quad P_1 \equiv (x_1, y_1, z_1), \]

where

\[
\begin{align*}
    x_1 &= \gamma, \
    y_1 &= \frac{\alpha \gamma}{1 - 2\gamma}, \
    z_1 &= \gamma \left( 1 + \frac{\alpha^2}{(1 - 2\gamma)^2} \right).
\end{align*}
\]

The Jacobian matrix for the system (1), denoted by \( J \), is given by

\[
J = \begin{pmatrix}
1 & \alpha + 4y & -1 \\
-\alpha - 2y & 1 - 2x & 0 \\
2z & 0 & 2(x - \gamma)
\end{pmatrix}.
\]
and the eigenvalue equation, calculated using $\det(J - \lambda I) = 0$, where $I$ is the $3 \times 3$ identity matrix, is

$$p(\lambda) = A\lambda^3 + B\lambda^2 + C\lambda + D = 0,$$  

(3)

where

$$A = 1, \quad B = 2(\gamma - 1),$$

$$C = -4x^2 + 2(2\gamma + 1)x + 8y^2 + 6\alpha y + 2z + \alpha^2 - 4\gamma + 1,$$

$$D = 4x^2 + (-16y^2 - 12\alpha y + 4z - 2\alpha^2 - 4\gamma - 2)x + 16\gamma y^2 + 12\alpha \gamma y - 2z + 2(\alpha^2 + 1)\gamma.$$

According to Eq. (3), the eigenvalue equation at the origin $P_0$ can be written as

$$p_0(\lambda) = \lambda^3 + 2(\gamma - 1)\lambda^2 + (\alpha^2 - 4\gamma + 1)\lambda + 2(\alpha^2 + 1)\gamma = 0,$$

from where results $\lambda_1 = -2\gamma$, $\lambda_2 = 1 + \alpha i$, $\lambda_3 = 1 - \alpha i$, with $i = \sqrt{-1}$, as the eigenvalues. The origin $P_0$ is a stable equilibrium point if the real part of all corresponding eigenvalues is negative, condition not satisfied. Therefore, $P_0$ is always unstable, and we conclude that chaos and limit cycles can be displayed by system (1), depending on parameters $\alpha$ and $\gamma$.

Again according to Eq. (3), the eigenvalue equation at $P_1$ is

$$p_1(\lambda) = \lambda^3 + 2(\gamma - 1)\lambda^2 + \frac{4\gamma^2 + 4\gamma(\alpha^2 - 1) + \alpha^2 + 1}{(2\gamma - 1)^2} \lambda + \frac{2(4\gamma^2 - 4\gamma + \alpha^2 + 1)\gamma}{2\gamma - 1} = 0.$$  

(4)

The solution of this last equation is straightforward and, consequently, the three eigenvalues associated with $P_1$ can be easily obtained. However, due to the complexity of the resulting three eigenvalue expressions, conclusions are difficult. Indeed, we are interested only in to determine the sign of the real part of the eigenvalues, which tell us about the stability of the equilibrium point $P_1$. With this purpose, we use the Routh-Hurwitz criterion [13]. First we construct the Routh table associated with the polynomial $p_1(\lambda)$ in (4), which is given by

$$
\begin{array}{ccc}
1 & 4\gamma^2 + 4\gamma(\alpha^2 - 1) + \alpha^2 + 1 & 2(4\gamma^2 - 4\gamma + \alpha^2 + 1)\gamma \\
2(\gamma - 1) & (2\gamma - 1)^2(2\gamma - 2\gamma - 1)\alpha^2 & 2\gamma - 1 \\
-(2\gamma^2 - 2\gamma + 1)(2\gamma - 1)^2(2\gamma - 2\gamma - 1)\alpha^2 & 0 & 0 \\
\end{array}
$$

All the roots of the polynomial $p_1(\lambda)$, i.e., all the eigenvalues associated with the equilibrium point $P_1$, have real part less than zero, if all four elements in the first column of the Routh table are nonzero and have the same sign. This condition leads to

$$\alpha^2 > \frac{(2\gamma^2 - 2\gamma + 1)(2\gamma - 1)^2}{(2\gamma^2 - 2\gamma - 1)}.$$  

(5)

From the inequality $\gamma > 1$ previously obtained, we conclude that the numerator of (5) is always positive. Hence, to ensure that $\alpha^2 > 0$, an additional restriction on $\gamma$ is necessary, namely $\gamma > (1 + \sqrt{3})/2$. Therefore, the parameter-space region where $P_1$ remains stable is defined by (5), with the condition $\gamma > (1 + \sqrt{3})/2$.

### 4 Numerical Simulations

Here we present various diagrams displaying the dynamic behavior of system (1), as the parameters $\gamma$ and $\alpha$ are varied. Figure 1(a) displays a typical parameter-space plot for system (1), obtained by computing the largest Lyapunov exponent on a mesh of $500 \times 500$ parameters $(\gamma, \alpha)$, by using the Wolf algorithm [14].

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Figure 1: (a) Regions and colors in $(\gamma, \alpha)$ parameter-space of system (1). Boxed regions are shown magnified in further figures. (b) Bifurcation diagram for points along the red line $\alpha = 0.4162\gamma - 0.5210$ in plot (a), showing the number of maxima of $x(t)$ variable. Numbers are referent to the quantity of maxima in $x(t)$ variable.

It shows a global view of the parameter-space $(\gamma, \alpha)$, for $10 \leq \gamma \leq 200$, $1 \leq \alpha \leq 100$, where we can see clearly details of an organization of periodicity domains in black, the shrimps labeled as 3, 4, 5, and 6, hereafter named periodic structures, embedded in a large chaotic region in yellow-red. Therefore, there are regions in the parameter-space where windows of periodic regions are separated by chaotic regions. In other words, as the parameters are varied, we observe a stable periodic structure, followed by a chaos region. Then we observe another periodic structure, also followed by a chaos region, and so on. Note that these periodic structures appear organized themselves along the red line $\alpha = 0.4162\gamma - 0.5210$ in Fig. 1(a), and constitute a piece of a period-adding bifurcation cascade which increases the periodicity by the unity, as $\gamma$ (or $\alpha$) increases. In these and further parameter-space plots, system (1) was always integrated with a fourth-order Runge-Kutta algorithm, with a fixed step size equal to $10^{-3}$, and $5 \times 10^{5}$ steps were considered to compute the largest Lyapunov exponent. To construct Fig. 1(a), integrations were started at $(\gamma, \alpha)=(10,1)$, from the initial condition $(x_0,y_0,z_0)=(5.0,5.0,0.1)$, and for each increment in $\gamma$ and/or $\alpha$, we follow the attractor, that is, we use the last obtained values for $(x,y,z)$ before the increment, as the new initial conditions after the increment. Colors are associated with the magnitude of the largest Lyapunov exponent, as shown in the column at right side in Fig. 1(a). White for more negative, black for zero, and red for more positive. Indeed, a positive exponent is indicated by a continuously changing yellow-red
Figure 1(b) shows a bifurcation diagram obtained when walking along the same red line $\alpha = 0.4162\gamma - 0.5210$ in Fig. 1(a). It was constructed by following the behavior of successive maxima of $x(t)$ variable, as a function of the bifurcation parameter $\gamma$. It can be seen from this plot, the typical alternation of chaos and periodicity, which is a characteristic of period-adding bifurcation cascades. Numbers placed in periodic windows of Fig. 1(b) and periodic structures in Fig. 1(a) refer to the number of maxima in $x(t)$ variable present in one period. Again system (1) was integrated with a fourth-order Runge-Kutta algorithm with a fixed step size equal to $10^{-3}$. The $x$ axis was divided in 500 values, the computation was started at $\gamma = 10$ from the same initial condition $(x_0, y_0, z_0)$ revealed above, and the attractor was followed. Note that the number of maxima in $x(t)$ decreases as $\gamma$ decreases, by an amount equal to the unity, which is exactly the number of maxima of the leftmost periodic region $5 \lesssim \gamma \lesssim 8$ not shown in the scale of plots in Fig. 1, to where the period-adding bifurcation cascade $\ldots 5 \rightarrow 4 \rightarrow 3 \rightarrow 2$ accumulates.

Figure 2 displays several plots of parameter-spaces, some of them showing magnification of regions inside the boxes in Fig. 1(a). Figure 2(a), for instance, shows magnification of the region inside the box A in Fig. 1(a). Period-doubling bifurcations occur as we approach to the boundary with chaotic region in yellow-red, independently of the followed
direction. Therefore, system (1) contains a period-doubling bifurcation cascade, a typical route to chaos, as $\gamma$ increases, from each point in large black region at left in Fig 2(a). Several series of periodic structures can be seen embedded in the chaotic region, and we marked the more prominent structure in three of them as 3, 5, and 6, which are the respective number of maxima in $x(t)$. It is apparent that here does not exist regularity in the organization of the periodic structures, different from that observed in Fig. 1(a). This last conclusion was evidenced by the construction of various bifurcation diagrams not shown here, for points along straight lines $\alpha = f(\gamma)$ aligned with each one of the series of periodic structures present in Fig 2(a).

Figures 2(b) and 2(d) show magnification of the regions inside the boxes B and D in Fig. 1(a), which are the regions among periodic structures $\{3,4\}$ and $\{5,6\}$, respectively. Here we detect some organization in periodic structures. In Fig. 2(b) we observe that the bigger structure is labeled with number 5, and is placed among structures 3 and 4 in Fig. 1(a), while in Fig. 2(d) the bigger structure is labeled with number 7, and is located among structures 5 and 6 in Fig. 1(a). This pattern of self-organization of the periodic structures embedded in the chaotic region of system (1) takes place at various levels, as can be seen in Fig. 2(b) where, for instance, a periodic structure labeled as 8 is the bigger that appears among structures 6 and 7. By combining Figs. 1(a) and 2(d) we can write the infinite sequence $5 \rightarrow 9 \rightarrow 8 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \rightarrow 11 \rightarrow \ldots 6$, which is followed by the maxima in $x(t)$ variable of some periodic structures embedded in a chaotic region. This behavior is recurrent in another scales describing, therefore, a kind of self-similarity.

Magnification of the periodic structure labeled as 5 inside the box C in Fig 1(a) is shown in Fig. 2(c). It is an example that represents the general form of all periodic structures. Figures 2(e) and 2(f) display magnification of the boxed regions A and B in Fig. 2(a), respectively. In the first we can observe a piece of the period-adding cascade ... $6 \rightarrow 8 \rightarrow 10 \rightarrow \ldots$ which increases the periodicity by a factor equal to 2, while in the second are shown two series of periodic structures symmetrically placed with regard to large black region labeled as 4, characterized both by a bigger structure labeled as 12. Figure 2(e) shows details of the upper right side in Fig. 2 of Ref. [12], where the alternation between chaos and periodicity was called by the authors intricate sequence of strange attractors and cycles.

5 Summary

In this paper we have investigated dynamical behaviors in a set of three autonomous first-order nonlinear ordinary differential equations, that model a nonlinear resonant coupling process which considers the nonlinear decay of a linearly unstable high-frequency wave into a linearly damped low-frequency wave. We have determined analytically the location of the equilibrium points, and the respective stability regions in parameter-space. We have shown numerically that the parameter-space of the studied system presents self-similar stable periodic structures embedded in a large chaotic region. We also have shown that these structures organize themselves in period-adding bifurcation cascades.

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References


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