Viscosity Approximation Methods for Fixed Points of Asymptotically Nonexpansive Semigroup in Banach Space

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Abstract: In this paper, under the framework of Banach space with uniformly Gateaux differentiable norm and uniform normal structure, we use the existence theorem of fixed points of Gang Li and Sims to investigate the convergence of the implicit iteration process and the explicit iteration process for asymptotically nonexpansive semigroup. We get the convergence theorems.

Keywords: Viscosity approximation methods; fixed point; asymptotically nonexpansive semigroups; uniform normal structure; uniformly Gateaux differentiable norm


1 Introduction

Let $E$ be a real Banach space, $E^*$ is the dual space of $E$, $K$ is a nonempty closed convex subset of $E$. Let $J : E → 2^{E^*}$ denote the normalized duality mapping defined by $J(x) := \{ f ∈ E^* : ⟨ f, x ⟩ = ∥ x ∥^2, ∥ f ∥ = ∥ x ∥, x ∈ E \}$. A mapping $T : K → K$ is called a contraction if there exists a constant $α ∈ [0, 1)$ such that $∥Tx − Ty∥ ≤ α ∥x − y∥, \forall x, y ∈ K$. A family $\{T(t) : t ≥ 0\}$ of mappings from $K$ into itself is said to be a one-parameter asymptotically nonexpansive semigroup on $K$ with Lipschitz constants $\{k(t) : t ≥ 0\}$ if the following are satisfied (cf. [16]):

(i) $∥T(t)x − T(t)y∥ ≤ k(t)∥x − y∥$ for all $x, y ∈ K$;
(ii) $T(t + s)x = T(t)T(s)x$ for all $t, s ≥ 0$ and $x ∈ K$;
(iii) $T(0)x = x$ for all $x ∈ K$;
(iv) for each $x ∈ K$, the mapping $t → T(t)x$ is continuous;
(v) $t → k(t) ; [0, ∞) → [0, ∞)$ is continuous;
(vi) $k(t) ≥ 1$ for all $t ≥ 0$ and $\limsup_{t→∞} k(t) = 1$.

Such a semigroup $\{T(t) : t ≥ 0\}$ is called a one-parameter asymptotically nonexpansive semigroup on $K$ if $k(t) = 1$ for all $t ≥ 0$. Let $F(3) = \bigcap_{t>0} Fix(T(t)) = \{ x ∈ K : T(t)x = x, t > 0 \}$.

In [1], Moudafi had proposed a viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. In [2], Xu studied the viscosity approximation methods proposed by Moudafi [1] for a nonexpansive mapping in a uniformly smooth Banach space.

Recently, N.Shahzad and A.Udomene [3] studied the convergence of the implicit iteration process and the explicit iteration process in a real Banach space with uniformly Gateaux differentiable norm and uniform normal structure.

In this paper, motivated by N.Shahzad and A.Udomene [3], we prove the following fixed point convergence theorems: Let $E$ be a Banach space with a uniformly Gateaux differentiable norm and uniform normal structure. Let $K$ be a nonempty bounded closed convex subset of $E$ and $\{T(t) : t ≥ 0\}$ be a one-parameter asymptotically nonexpansive semigroup on $K$ with Lipschitz constants $\{k(t) : t ≥ 0\}$ satisfies $\lim_{n→∞} α_n = 0$, at the same time $lim_{t→∞} t_n = \infty$. Then for any $n ∈ N$, there exist an integer $l(n)$ and a unique $x_n ∈ K$ such that $x_n = α_n f(x_n) + (1 − α_n)T(t_l(n))x_n$. Further, (1) if $\lim_{n→∞} ∥x_n − T(t)x_n∥ = 0$ for all $t ≥ 0$, then $\{x_n\}$ converges strongly to a fixed point $p ∈ F(3)$, which is also the unique solution to the variational

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inequality: \((I - f)p, j(p - x^*) \leq 0, \forall x^* \in F(\mathcal{S})\). If \(\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty\) and for any \(y_0 \in K\), the explicit iteration process \(y_{n+1} := \alpha_n f(y_n) + (1 - \alpha_n) T(t_n)y_n, n \geq 1\), satisfies \(\lim_n \|T(t)y_n - y_n\| = 0\) for all \(t \geq 0\), then \(\{y_n\}\) converges strongly to a fixed point \(p \in F(\mathcal{S})\).

Our main results extend the theorems in [2,3,4] to the class of asymptotically nonexpansive semigroup, and remove some key conditions of iterative coefficient (cf.[2,3,4,5,6,7,8,17,18,19,20,21]).

2 Preliminaries

Let \(S := \{x \in E : \|x\| = 1\}\) denote the unit sphere of the Banach space \(E\). The space \(E\) is said to have a Gateaux differentiable norm if the limit

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for each \(x, y \in S\); and \(E\) is said to have a uniformly Gateaux differentiable norm if for each \(y \in S\) the limit (*) is attained uniformly for \(x \in S\). Further, \(E\) is said to be uniformly smooth if the limit (*) exists uniformly for \((x, y) \in S \times S\).

It is well known (9) that if \(E\) is smooth then any duality mapping on \(E\) is single-valued, and if \(E\) has a uniformly Gateaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded sets.

Let \(K\) be a nonempty closed convex and bounded subset of the Banach space \(E\) and let \(r(x, \mathcal{K}) := \sup \{\|x - y\| : y \in \mathcal{K}\}\) for each \(x \in K\), let \(r(x, K) := \sup \{\|x - y\| : y \in K\}\) and let \(r(K) := \inf \{r(x, K) : x \in K\}\) denote the Chebyshev radius of \(K\) relative to itself. The normal structure coefficient \(N(E)\) of \(E\) (cf.[10]) is defined by

\[
N(E) := \inf \{d(K)/r(K) : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0\}.
\]

A space \(E\) such that \(N(E) > 1\) is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g., [11,12]).

We shall let \(\text{LIM}\) be a Banach limit. Recall that \(\text{LIM} \in (\ell^\infty)^*\). such that \(\|\text{LIM}\| = 1, \lim_{n \to \infty} \inf a_n \leq \text{LIM} a_n \leq \lim_{n \to \infty} \sup a_n\), and \(\text{LIM} a_n = \text{LIM} a_n+1\) for all \(\{a_n\}_n \in \ell^\infty\).

The following lemmas will be needed.

**Lemma 1** (Chidume[13] lemma1, Xu[2] lemma2) Let \(E\) be an arbitrary real Banach space. Then

\[
\forall x, y \in E, \forall j \in J(x + y) \implies \|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle
\]

**Lemma 2** (Naseer Shahzad, Aniefiok Udomene [3], lemma2) Let \(E\) be a Banach space with uniform normal structure, \(K\) a nonempty closed convex and bounded subset of \(E\), and \(T : K \to K\) an asymptotically nonexpansive mapping. Then \(T\) has a fixed point.

**Lemma 3** (Chidume[13] lemma SR) Let \(K\) be a nonempty closed convex subset of a Banach space \(E\) with a uniformly Gateaux differentiable norm and let \(\{x_n\}\) be a bounded sequence in \(K\). Let \(g(x) = \text{LIM} \|x_n - x\|^2, x \in E\) be a Banach limit and \(z \in C\). Then

\[
g(z) = \min_{x \in K} g(x)
\]

if and only if

\[
\text{LIM}_{n \to \infty} \langle y - z, J(x_n - z) \rangle \leq 0,
\]

for all \(\forall y \in K\).

**Lemma 4** (Xu[2]) Assume \(\{a_n\}\) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \beta_n, n = 0, 1, 2, \cdots
\]

where \(\{\gamma_n\} \subset [0, 1], \{\beta_n\} \subset [0, 1]\) such that \(\sum_{n=0}^{\infty} \gamma_n = \infty, \lim_{n \to \infty} \beta_n = 0\), then \(\lim_{n \to \infty} a_n = 0\).
Lemma 5 (Gang Li-Brailey Sims[15], Theorem 2.1) Suppose X is a Banach space with uniform normal structure; C is a nonempty bounded subset of X; and $\Theta = \{ T(t) : t \geq 0 \}$ is a semigroup of asymptotically nonexpansive type mappings on C such that $T(t)$ is continuous on C for each $t \geq 0$. Further, suppose that there exists a nonempty closed convex subset $E$ of C with the following property (P):

$$x \in E \implies \omega_{w}(x) \subset E;$$

where $\omega_{w}(x)$ is the weak $\omega$-limit set of $\{ T(t)x \}$; that is, the set

$$\{ y \in X : y = \text{weak} - \lim_{i} T(t_{i})x \text{ for some } t_{i} \uparrow \infty \}.$$

Then $\Theta$ has a common fixed point in $E$, i.e. there exists a $z \in E$ for which $T(t)z = z$ for all $t \geq 0$.

### 3 Main results

**Theorem 6** Let $E$ be a Banach space with a uniformly Gateaux differentiable norm and uniform normal structure. Let $K$ be a nonempty bounded closed convex subset of $E$ and $\{ T(t) : t \geq 0 \}$ be an asymptotically nonexpansive semigroup on $K$. If $f : K \to K$ is a contraction and a sequence $\{ \alpha_{n} \} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_{n} = 0,$ $\{ t_{n} \} \subset (0, +\infty)$ and $\lim_{n \to \infty} t_{n} = \infty$.

Then

(i) for any $n \in N$, there exist an integer $l (n)$ and a unique $x_{n} \in K$ such that

$$x_{n} = \alpha_{n} f (x_{n}) + (1 - \alpha_{n}) T(t_{l(n)}) x_{n}.$$  \hspace{1cm} (1)

(ii) if $\lim_{n \to \infty} \| x_{n} - T(t)x_{n} \| = 0$ for all fixed $t \geq 0$, then $\{ x_{n} \}$ converges strongly to a fixed point $p \in F(\Theta)$, which is also the unique solution to the variational inequality:

$$\langle (I - f)p, j(p - x) \rangle \leq 0, \forall x \in F(\Theta).$$  \hspace{1cm} (2)

**Proof.** (i) For each $\alpha_{n} \in (0, 1),$ $\lim_{t \to \infty} k_{t} = 1$, then there exist an integer $l (n) > 0$ such that $k_{l(n)} - 1 < \alpha_{n}^{2}, 0 \leq \lim_{n \to \infty} \frac{k_{l(n) - 1}}{\alpha_{n}} \leq \lim_{n \to \infty} \frac{\alpha_{n}^{2}}{\alpha_{n}^{n}} = 0$, this implies that

$$k_{l(n)} - 1 = o(\alpha_{n}) (n \to \infty).$$

By the conditions on $\{ \alpha_{n} \}$, for each integer $n \geq 0$, the mapping $S_{n} : K \to K$ defined for each $x \in K$ by $S_{n} (x) := \alpha_{n} f (x_{n}) + (1 - \alpha_{n}) T(t_{l(n)}) x_{n}$ is a contraction. When $n$ is large enough, in fact, $\forall x, y \in K$, let $k_{l(n)} = k_{l(n)}$, then

$$\| S_{n} (x) - S_{n} (y) \| \leq \alpha_{n} \| f (x) - f (y) \| + (1 - \alpha_{n}) \| T(t_{l(n)}) x - T(t_{l(n)}) y \|$$

$$\leq [ \alpha \cdot \alpha_{n} + (1 - \alpha_{n}) k_{l(n)} ] \cdot \| x - y \|.$$  \hspace{1cm} (3)

As $k_{l(n)} = 1 = o(\alpha_{n}) (n \to \infty)$, there exist $n_{0} \in N$, when $n \geq n_{0}$ and there must be $\frac{k_{l(n) - 1}}{\alpha_{n}} < 1 - \alpha < \frac{1 - \alpha}{1 - \alpha_{n}}$, that is $\alpha \cdot \alpha_{n} + (1 - \alpha_{n}) k_{l(n)} < 1$, so $S_{n}$ is a contraction. It follows that there exists a unique $x_{n} \in K$ such that $S_{n} x_{n} = x_{n}$.

(ii) First, we show the uniqueness of solutions of the variational inequality(2). In fact, supposing $p, q \in F(\Theta)$ satisfy (2), we get that

$$\langle (I - f)p, j(p - q) \rangle \leq 0.$$  \hspace{1cm} (3)

$$\langle (I - f)q, j(q - p) \rangle \leq 0.$$  \hspace{1cm} (4)

Adding up (3) and (4), we have that

$$(1 - \alpha) \| p - q \|^{2} \leq \langle (I - f)p - (I - f)q, j(p - q) \rangle \leq 0,$$

We must have $p = q$ and the uniqueness is proved.
Define the mapping \( g : K \to R \) by \( g(x) := LIM_n \| x_n - x \|^2, \forall x \in E. \) Then since the function \( g \) on \( K \) is convex and continuous, \( g(x) \to \infty \) as \( \|x\| \to \infty \), and \( E \) is reflexive, there exists \( z \in K \) with \( g(z) = \inf_{x \in K} g(x) \neq 0 \), and \( M \) is a nonempty closed convex subset of \( K \).

Now, we prove that there exists \( p \in M \) such that \( p \in F(3) \).

\[ \forall z \in M, K \text{ is bounded, and so are } \{ T(t_{i(n)}) z \}. \] It is known that every space with a uniform normal structure is reflexive, so \( E \) is reflexive, there exists a weakly convergent subsequence \( \{ T(t_{i(n)}) z \} \), and now we suppose \( T(t_{i(n)}) z \to z_0 (i \to \infty) \).

For any one fixed integer \( s > 0 \), we have

\[ \| x_n - T(s)z \| \leq \| x_n - T(s)x_n \| + k_s \| x_n - z \|, \] (5)

Using \( \lim_{n \to \infty} \| x_n - T(t)x_n \| = 0 \), we obtain \( \lim_{n \to \infty} \| x_n - T(s)x_n \| = 0 \).

Since

\[ g(T(s)z) = LIM_n \| x_n - T(s)z \| \leq LIM_n \| x_n - T(s)x_n \| + LIM_n k_s \| x_n - z \| \leq k_s g(z) \]

Hence

\[ g(z_0) = \lim_{i \to \infty} g(T(t_{i(n)}) z) \leq \lim_{i \to \infty} k_s g(z) = g(z), \]

Now we have \( z_0 \in M \).

That is \( \forall z \in M, w_w(z) \subset M \) from above. By lemma 5, we conclude that there exists \( p \in M \) such that \( p \in F(3) \).

Now we show that \( \{ x_n \} \) is relatively sequentially compact.

Indeed, let \( N_1 \) is an infinite subset of \( N \), then \( \{ y_n : n \in N_1 \} \) is a subsequence of \( \{ x_n : n \in N \} \). Define the mapping \( g : K \to R \) by \( \tilde{g}(x) := LIM_n \| y_n - x \|^2, \forall x \in K \), we have \( \tilde{M} = \{ z : \tilde{g}(z) = \min_{x \in K} g(x) \} \neq \emptyset \), and \( \tilde{M} \cap F(3) \neq \emptyset \). \( \forall p \in \tilde{M} \cap F(3) \), from the iterative process (1), we estimate as follows:

\[ \langle y_n - f(y_n), j(y_n - p) \rangle = \frac{1 - \alpha_n}{\alpha_n} \langle T(t_{i(n)}) y_n - y_n, j(y_n - p) \rangle \]

\[ = \frac{1 - \alpha_n}{\alpha_n} \langle T(t_{i(n)}) y_n - y_n - (y_n - p), j(y_n - p) \rangle \]

\[ \leq \frac{1 - \alpha_n}{\alpha_n} (k_{i(n)} - 1) \| y_n - p \|^2, \]

So that

\[ LIM_n \langle y_n - f(y_n), j(y_n - p) \rangle \leq 0, \] (6)

by \( p \in \tilde{M} \) and lemma 3, we have

\[ LIM_n \langle y - p, j(y_n - p) \rangle \leq 0, \forall y \in K. \] (7)

In Eq.(7), let \( y = f(p) \) and add up to Eq.(6), then

\[ LIM_n \langle y_n - f(y_n) + f(p) - p, j(y_n - p) \rangle \leq 0, \]

We have

\[ LIM_n \| y_n - p \|^2 \leq LIM_n \langle f(y_n) - f(p), j(y_n - p) \rangle \leq \alpha LIM_n \| y_n - p \|^2, \]

i.e. \( LIM_n \| y_n - p \|^2 = 0 \), hence, there exists a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) which strongly converges to \( p \).

Finally, we show that \( x_n \to p \).

Let a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) strongly converges to \( p \), we have \( \lim_{n_j \to \infty} \| x_{n_j} - T(t)x_{n_j} \| = 0 \) because of

\[ \lim_{n \to \infty} \| x_n - T(t)x_n \| = 0 \]
so that \( \| q - T(t)q \| = 0 \), i.e. \( q \in F(\mathcal{S}) \).

From Eq. (6), we get
\[
\langle q - f(q), j(q - p) \rangle \leq 0, \forall p \in F(\mathcal{S}).
\]

This \( q \in F(\mathcal{S}) \) is the solution of the variational inequality (2); hence \( p = q \) by uniqueness. In summary, we have proved that \( \{ x_n \} \) is relatively sequentially compact and each cluster point of \( \{ x_n \} \) (as \( n \to \infty \)) equals \( q \). Therefore \( x_n \to q \) as \( n \to \infty \).

**Theorem 7** Let \( E \) be a Banach space with a uniformly Gateaux differentiable norm and uniform normal structure. Let \( K \) be a nonempty bounded closed convex subset of \( E \) and \( \{ T(t), t \geq 0 \} \) be an asymptotically nonexpansive semigroup on \( K \). If \( f : K \to K \) is a contraction and a sequence \( \{ \alpha_n \} \subset (0,1) \) satifies \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty \). At the same time \( \{ t_n \} \subset (0, +\infty) \) and \( \lim_{n \to \infty} t_n = \infty \). Then

(i) for any \( n \in N \), there exist an integer \( l(n) \) and a unique \( x_n \in K \) such that
\[
x_n = \alpha_n f(x_n) + (1 - \alpha_n) T(t_{l(n)}) x_n.
\]

(ii) if \( \lim_{n \to \infty} \| x_n - T(t) x_n \| = 0 \) for all fixed \( t \geq 0 \) and for any \( y_0 \in K \), the explicit iteration process
\[
y_{n+1} := \alpha_n f(y_n) + (1 - \alpha_n) T(t_{l(n)}) y_n, y_0 \in K, n \geq 1,
\]
satisfies \( \lim_{n \to \infty} \| y_n - T(t) y_n \| = 0 \) for all fixed \( t \geq 0 \), then \( \{ y_n \} \) converges strongly to a fixed point \( p \in F(\mathcal{S}) \), which is also the unique solution to the variational inequality (2).

**Proof.** Part (i) has already been proved in Theorem 6. Assume that \( \lim_{n \to \infty} \| x_n - T(t) x_n \| = 0 \) and \( \lim_{n \to \infty} \| y_n - T(t) y_n \| = 0 \) for all fixed \( t \geq 0 \). We proceed to prove part (ii). Let \( n > m \). Then, from (1) and lemma 1 we have
\[
\| x_m - y_n \|^2 \leq \| T(t_{l(m)}) x_m - y_n \|^2 + 2\alpha_m \langle f(x_m) - T(t_{l(m)}) x_m, j(x_m - y_n) \rangle,
\]
so that
\[
\langle f(x_m) - T(t_{l(m)}) x_m, j(y_n - x_m) \rangle \leq \frac{1}{2\alpha_m} \left( \| T(t_{l(m)}) x_m - y_n \|^2 - \| x_m - y_n \|^2 \right)
\]
\[
\leq \frac{1}{2\alpha_m} \left[ \left( \| T(t_{l(m)}) x_m - T(t_{l(m)}) y_n \| + \| T(t_{l(m)}) y_n - y_n \| \right)^2 - \| x_m - y_n \|^2 \right]
\]
\[
\leq \frac{1}{2\alpha_m} \left[ \left( \frac{k_{l(m)}^2}{2} - 1 \right) \| x_m - y_n \|^2 + \| T(t_{l(m)}) y_n - y_n \| \cdot 2k_{l(m)} \| x_m - y_n \| + \| T(t_{l(m)}) y_n - y_n \| \right).
\]

Since \( K \) is bounded, for some constant \( M > 0 \) such that
\[
\max \left\{ \| x_m - y_n \|^2, 2k_{l(m)} \| x_m - y_n \| + \| T(t_{l(m)}) y_n - y_n \| \right\} \leq M,
\]
it follows that
\[
\langle f(x_m) - T(t_{l(m)}) x_m, j(y_n - x_m) \rangle \leq \frac{k_{l(m)}^2}{2\alpha_m} - 1 M + \frac{1}{2\alpha_m} \| T(t_{l(m)}) y_n - y_n \| M,
\]
so that
\[
\lim_{n \to \infty} \sup_{m \to \infty} \langle f(x_m) - T(t_{l(m)}) x_m, j(y_n - x_m) \rangle \leq \frac{k_{l(m)}^2}{2\alpha_m} - 1 M.
\]

By Theorem 6, \( x_m \to p \in F(\mathcal{S}) \), which solve the variational inequality (2). Since \( j \) is norm to weak* continuous on bounded sets, in the limit as \( m \to \infty \), we obtain that
\[
\lim_{n \to \infty} \sup_{t} (f(p) - p, y_n - p) \leq 0
\]  

(9)

From Eq.(9), we observe that, there exists a sequence \( \{\varepsilon_n\} \), \( \varepsilon_n \geq 0 \) for all \( n \geq 0 \) such that \( \langle f(p) - p, j(y_n - p) \rangle \leq \varepsilon_n \) with \( \varepsilon_n \to 0 \) as \( n \to \infty \). Now, from the iterative process (8) and Lemma 1, we estimate as follows: \( \forall p \in F(3) \)

\[
\|y_{n+1} - p\|^2 \leq \|T(t_{i(n)}) y_n - p\|^2 + 2\alpha_n \langle f(y_n) - T(t_{i(n)}) y_n, j(y_{n+1} - p) \rangle
\]

\[
= \|T(t_{i(n)}) y_n - p\|^2 + 2\alpha_n \langle f(y_n) - f(p) + f(p) - p + y_{n+1} + y_{n+1} - T(t_{i(n)}) y_n, j(y_{n+1} - p) \rangle
\]

\[
\leq \|T(t_{i(n)}) y_n - p\|^2 + 2\alpha_n \left[ \alpha \|y_n - p\| \cdot \|y_{n+1} - p\| - \|y_{n+1} - p\|^2 \right]
\]

\[
+ 2\alpha_n \left[ \|y_{n+1} - T(t_{i(n)}) y_n\| \cdot \|y_{n+1} - p\| + \varepsilon_n \right].
\]

Since \( K \) is bounded, for some constant \( d > 0 \) such that \( \|f(x) - T(t_{i(n)}) y_n\| \leq d \), accordingly to (8) we have

\[
\|y_{n+1} - T(t_{i(n)}) y_n\| = \alpha_n \|f(x) - T(t_{i(n)}) y_n\| \leq \alpha_n d,
\]

So that

\[
\|y_{n+1} - p\|^2 \leq k_{i(n)}^2 \|y_n - p\|^2 + 2\alpha_n \left[ \alpha \|y_n - p\| \cdot \|y_{n+1} - p\| - \|y_{n+1} - p\|^2 + \alpha_n d \|y_{n+1} - p\| + \varepsilon_n \right]
\]

\[
\leq k_{i(n)}^2 \|y_n - p\|^2 + \alpha_n \left[ \alpha \left( \|y_n - p\|^2 + \|y_{n+1} - p\|^2 \right) - 2 \|y_{n+1} - p\|^2 + \alpha_n d \left( 1 + \|y_{n+1} - p\|^2 \right) + 2\varepsilon_n \right].
\]

Let

\[
\|y_n - p\|^2 = a_n, \lambda_n = \frac{1 - 2\alpha \cdot \alpha_n + 2\alpha_n - \alpha_n^2 \cdot d - k_{i(n)}^2}{1 - \alpha \cdot \alpha_n + 2\alpha_n - \alpha_n^2}, \gamma_n = \frac{2\alpha_n \cdot \varepsilon_n + \alpha_n^2 \cdot d}{1 - \alpha \cdot \alpha_n + 2\alpha_n - \alpha_n^2 \cdot d},
\]

we get

\[
a_{n+1} \leq (1 - \lambda_n) a_n + \gamma_n.
\]

It is clear that \( \lim_{n \to \infty} \lambda_n = 2 - 2\alpha > 0 \), then \( \lim_{n \to \infty} \frac{\lambda_n}{\alpha_n(1 - \alpha_n)} = 2 - 2\alpha > 0 \), it can easily be show that \( \sum_{n=0}^{\infty} \lambda_n = \infty \) by \( \sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty \). Furthermore, \( \lim_{n \to \infty} \frac{\lambda_n}{\alpha_n} = 0 \). Hence, it follows from Lemma 4 that \( \lim_{n \to \infty} a_n = 0 \), i.e. \( \lim_{n \to \infty} \|y_n - p\| = 0 \), then \( \{y_n\} \) converges strongly to a fixed point \( p \in F(3) \), which is also the unique solution to the variational inequality (2).

References


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