Some Existence Results of Solutions for Fractional Initial Value Problem

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Abstract: In this paper, the existence results of solutions for fractional initial value problem involving Caputo
differential operators are discussed. As applications of Ascoli-Arzela’s theorem, we prove the local existence
result and consider the existence of extremal solutions. Using the comparison result, we discuss a global
existence result in the end.

Keywords: Fractional differential equations; Initial value problem; Comparison result

1 Introduction

Much attention has been paid to the existence of solutions for fractional equations due to its wide application in engi-
neering, economics and other fields. There has been significant development in the existence of solutions to initial value
problem for fractional differential equations, see [1-5]. The most research method is changing the initial value problem to
an equivalent Volterra integral equation, then using some fixed-point theorems, the existence and uniqueness of solutions
are obtained. However, few papers have considered the existence of extremal solutions and the global existence. Recently,
V. Lakshmikantham and A.S. Vatsala [6] has addressed this question, where the fractional order is \( 0 < q < 1 \). In this paper,
we discuss the existence of solutions for the initial value problem of fractional differential equations involving
Caputo\(^{[7,8]}\) differential operators of order \( 1 < \alpha < 2 \). We shall start with the fundamental theory of inequalities, which
provide necessary comparison results that are useful in further study of qualitative and quantitative properties of solutions
of fractional differential equations. We then prove Peano’s local existence result and consider the existence of extremal
solutions. As an application of the comparison result developed, we discuss a global existence result in the end.

2 Theory of Inequalities

Consider the initial value problem (IVP) for fractional differential equations given by

\[
\begin{align*}
D^\alpha x(t) &= f(t, x(t)), \\
x(0) &= x_0, x'(0) = y_0 
\end{align*}
\]

where \( f \in C([0, T], \mathbb{R}) \), \( D^\alpha x \) is the fractional derivative of \( x \) and \( \alpha \) is such that \( 1 < \alpha < 2 \). since \( f \) is assumed continuous,
the IVP (1) is equivalent to the following Volterra fractional integral

\[
x(t) = x_0 + y_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, x(\tau))d\tau
\]

Let us first discuss a fundamental result relative to the fractional integral inequalities.

**Theorem 1** Let \( v, w \in C^1([0, T], \mathbb{R}), f \in C([0, T], \mathbb{R}), \) and \( f(t, x) \) is nondecreasing in \( x \) for each \( t \) and

(i) \( v(t) \leq v(0) + v'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, v(\tau))d\tau \)

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If condition a satisfies, let us suppose that the inequality we get

\[ v(0) \leq w(0), v'(0) \leq w'(0) \]  

(3)

Then we have

\[ v(t) < w(t), 0 \leq t \leq T. \]  

(5)

**proof.** Suppose that the conclusion (5) is not true. Then, there exists a \( t_1 \in [0, T] \), and

\[ v(t_1) = w(t_1), v(t) < w(t), 0 < t \leq t_1, \]  

(6)

If condition a satisfies, let us suppose that the inequality (ii) is strict. Then using the nondecreasing nature of \( f \) and (6), we get

\[ w(t_1) > w(0) + w'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, w(\tau))d\tau + \epsilon(1 + t^\alpha) \]

\[ \geq v(0) + v'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, v(\tau))d\tau \]

\[ \geq v(t_1). \]

Which is a contradiction in view of (6), hence the conclusion (5) is valid and the proof is complete.

If condition b satisfies, using the nondecreasing nature of \( f \), we get

\[ w(t_1) \geq w(0) + w'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, w(\tau))d\tau \]

\[ \geq v(0) + v'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, v(\tau))d\tau \]

\[ \geq v(t_1). \]

Which is a contradiction in view of (6), hence the conclusion (5) is valid and the proof is complete.

The next result is for nonstrict inequalities, which requires a one-sided Lipschitz type condition.

**Theorem 2** Assume that the conditions of Theorem 1 hold with nonstrict inequalities (i) and (ii). Suppose further that

\[ f(t, x) - f(t, y) \leq \frac{L}{1 + t^\alpha}(x - y) \]

(7)

Whenever \( x \geq y, L > 0 \). If \( v(0) \leq w(0), v'(0) \leq w'(0), L < \Gamma(\alpha + 1) \), then we have

\[ v(t) \leq w(t), 0 \leq t \leq T. \]  

(8)

**proof.** Set \( w_\epsilon(t) = w(t) + \epsilon(1 + t^\alpha) \), for small \( \epsilon > 0 \), so that we have

\[ w_\epsilon(0) = w(0) + \epsilon > w(0), w'_\epsilon(0) = w'(0), w_\epsilon(t) > w(t) \]

(9)

In view of (9), using one-sided Lipschitz condition (7), we see that

\[ w_\epsilon(t) = w(t) + \epsilon(1 + t^\alpha) \]

\[ \geq w(0) + w'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, w(\tau))d\tau + \epsilon(1 + t^\alpha) \]

\[ \geq w_\epsilon(0) + w'_\epsilon(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, w_\epsilon(\tau))d\tau - \frac{\epsilon L}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau + \epsilon t^\alpha. \]

Because \( \int_0^t (t - \tau)^{\alpha-1} d\tau = \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} t^\alpha \), so

\[ - \frac{\epsilon L}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau + \epsilon t^\alpha = \epsilon t^\alpha(1 - \frac{L}{\Gamma(\alpha + 1)}) \geq 0. \]  

We arrive at

\[ w_\epsilon(t) \geq w_\epsilon(0) + w'_\epsilon(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, w_\epsilon(\tau))d\tau. \]

(10)

We now apply Theorem 1 to the inequalities (i), we get

\[ v(t) < w_\epsilon(t), 0 \leq t \leq T. \]  

Since \( \epsilon > 0 \) is arbitrary, we conclude that (8) is true.
3 The existence of local solutions and extremal solutions

In this section, we shall consider the local existence and the existence of extremal solutions for the IVP(1). Let us first discuss Peano’s type existence result.

**Theorem 3** Assume that \( f \in [R_0, R] \), where \( R_0 = [(t, x) : 0 \leq t \leq a, |x - x_0| \leq b, b > 2|y_0|T] \), and let \(|f(t, x)| \leq M\), then the VIP (1) possesses at least one solution \( x(t) \) on \( 0 \leq t \leq \beta \). Where \( \beta = \min(a, \frac{1}{\max(M, r(t, x))} [\Gamma(\alpha + 1)]^\frac{1}{2}) \), \( 1 < \alpha < 2, \Gamma \) being the Gamma function as before.

**proof.** Let \( x_0(t) \) be a continuous function on \([-\delta, 0], \delta > 0\), such that \( x_0(0) = x_0, x'_0(0) = y_0, |x_0(t) - x_0| \leq b \), for \( \forall \epsilon, (0 < \epsilon \leq \delta) \), we define a function \( x_\epsilon(t) \) on \([-\delta, 0]\), \( \beta_1 = \min(\beta, \epsilon) \), we observe that

\[
x_\epsilon(t) = \begin{cases} 
0, & t \in [-\delta, 0) \\
x_0(t) - x_\epsilon(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x_\epsilon(\tau - \epsilon)) d\tau, & t \in [0, \beta_1]
\end{cases}
\]

on \([0, \beta_1]\), where \( \beta_1 = \min(\beta, \epsilon) \), we observe that

\[
|x_\epsilon(t) - x_0| \leq |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^\beta (t - \tau)^{\alpha - 1} |f(\tau, x_\epsilon(\tau - \epsilon))| d\tau \leq |y_0| + \frac{M}{\Gamma(\alpha)} \int_0^\beta (t - \tau)^{\alpha - 1} d\tau \leq |y_0| + \frac{M}{\Gamma(\alpha + 1)} \beta \leq b.
\]

Because of the choice of \( \beta_1 \). If \( \beta_1 < \beta \), we can employ (12) to extend as a continuous function on \([-\delta, \beta_2], \beta_2 = \min(\beta, 2\epsilon) \), such that \( |x_\epsilon(t) - x_0| \leq b \) holds. Continuing this process, we can define \( x_\epsilon(t) \) over \([-\delta, \beta] \), so that \( |x_\epsilon(t) - x_0| \leq b \) is satisfied on \([-\delta, \beta] \). Furthermore, let \( 0 \leq t_1 \leq t_2 \leq \beta \), \( |t_2 - t_1| < \delta_0 = \frac{\epsilon}{2M + |y_0| \Gamma(\alpha + 1)} \), we see that

\[
|x_\epsilon(t_1) - x_\epsilon(t_2)| = |y_0(t_1 - t_2) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - \tau)^{\alpha - 1} f(\tau, x_\epsilon(\tau - \epsilon)) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - \tau)^{\alpha - 1} f(\tau, x_\epsilon(\tau - \epsilon)) d\tau| \\
\leq |y_0| |t_1 - t_2| + \frac{M}{\Gamma(\alpha)} |\int_0^{t_1} (t_1 - \tau)^{\alpha - 1} - (t_2 - \tau)^{\alpha - 1}| d\tau| \\
\leq |y_0| |t_1 - t_2| + \frac{M}{\Gamma(\alpha)} \beta |t_1 - t_2|^\alpha \leq |y_0| |t_1 - t_2|^{\alpha - 1} + \frac{M}{\Gamma(\alpha + 1)} |t_1 - t_2|^\alpha \\
\leq |y_0| |t_1 - t_2|^{\alpha - 1} + \frac{M}{\Gamma(\alpha + 1)} |t_1 - t_2|^\alpha.
\]

It then follows from (12) and (13) that the family \( \{x_\epsilon(t)\} \) forms an equicontinuous and uniformly bounded functions. As application of Ascoli-Arzela’s theorem shows the existence of a sequence \( \{\epsilon_n\} \) such that \( \epsilon_1 > \epsilon_2 > \cdots > \epsilon_n \to 0 \) as \( n \to \infty \), and \( x(t) = \lim_{n \to \infty} x_{\epsilon_n}(t) \) exists uniformly on \([-\delta, \beta] \). Since \( f \) is uniformly continuous, we obtain that \( f(t, x_{\epsilon_n}(t - \epsilon_n)) \) tends to uniformly to \( f(t, x(t)) \) as \( n \to \infty \). So we have

\[
x(t) = x_0 + y_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau)) d\tau.
\]

This proves that \( x(t) \) is a solution of the IVP (1) and the proof is complete.

Using Theorems 1 and 3, we can now prove the existence of extremal solutions for the IVP (1).

**Theorem 4** Under the hypothesis of Theorem 3, there exists extremal solutions for the IVP (1) on the interval \( 0 \leq t \leq \beta_0 \), \( \beta_0 = \min(a, \frac{1}{\max(M, r(t, x))} [\Gamma(\alpha + 1)]^\frac{1}{2}) \) provided \( f(t, x) \) is nondecreasing in \( x \) for each \( t \).

**proof.** We shall prove the existence of maximal solution only, since the case of minimal solution is very similar. Let \( 0 < \epsilon \leq \frac{b}{2} \) and consider the fractional differential equation with an initial condition

\[
\begin{cases} 
D^\alpha x(t) = f(t, x(t)) + \epsilon, \\
x(0) = x_0 + \epsilon, x'(0) = y_0
\end{cases}
\]

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We observe that \( f(t, x) = f(x, \epsilon) \) is defined and continuous on \( R_\epsilon = [0 \leq t \leq a, |x - (x_0 - \epsilon)| \leq \frac{b}{2} \), \( R_\epsilon \subset R_0 \), and \( |f(t, x)| \leq M + \frac{b}{2} \) on \( R_\epsilon \). We then deduce from Theorem 3 that the IVP (14) has a solution \( x(t, \epsilon) \) on the interval \([0, \beta_0]\). Now for \( 0 < \epsilon_2 < \epsilon_1 \leq \epsilon \), we have

\[
x(0, \epsilon_2) < x(0, \epsilon_1), \quad x'(0, \epsilon_2) < x'(0, \epsilon_1),
\]

\[
x(t, \epsilon_2) \leq x(0, \epsilon_2) + x'(0, \epsilon_2)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f_\epsilon(\tau, x_{\epsilon_2})d\tau,
\]

\[
x(t, \epsilon_1) \geq x(0, \epsilon_1) + x'(0, \epsilon_1)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} [f(\tau, x(\tau, \epsilon_1)) + \epsilon_2]d\tau.
\]

We apply Theorem 1 to get

\[
x(t, \epsilon_2) < x(t, \epsilon_1), \quad t \in [0, \beta_0].
\]

Consider the family of functions \( \{x(t, \epsilon) - x(0, \epsilon)\} \) on \([0, \beta_0]\), we have

\[
|\eta(t) - \eta(0)| \leq |y_0|T + |y_0| \int_0^t (t - \tau)^{\alpha - 1} |f(\tau, x(\tau, \epsilon))|d\tau
\]

\[
\leq |y_0|T + \frac{2M + b}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1}d\tau
\]

\[
\leq |y_0|T + \frac{2M + b}{\Gamma(\alpha)} T_0
\]

\[
\leq \frac{b}{2} < b.
\]

If \( 0 \leq t_1 \leq t_2 \leq \beta_0 \), then \( |x(t_1, \epsilon) - x(t_2, \epsilon)| \leq \left( \frac{2M + b}{\Gamma(\alpha)} + |y_0| \right)(t_2 - t_1)^\alpha \).

Following the computation similar to (13) with suitable changes. So we prove that the family \( \{x(t, \epsilon)\} \) is uniformly bounded and equicontinuous. Hence there exists a sequence \( \epsilon_n \) with \( \epsilon_n \to \infty \) as \( n \to \infty \) and the uniform limit

\[
\eta(t) = \lim_{n \to \infty} x(t, \epsilon_n)
\]

exists on \([0, \beta_0]\). Clearly \( \eta(0) = x_0, \eta'(0) = y_0 \). The uniform continuity of \( f \), gives argument as before, that \( \eta(t) \) is a solution of IVP (1).

Next we show that \( \eta(t) \) is the required maximal solution of (1) on \([0, \beta_0]\). Let \( x(t) \) be any solution of (1) on \([0, \beta_0]\). Then we have

\[
x_0 < x_0 + \epsilon = x(0, \epsilon), \quad y_0 \leq y_0 = x'(0, \epsilon),
\]

\[
x(t) \leq x_0 + y_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} [f(\tau, x(\tau)) + \epsilon]\tau d\tau,
\]

\[
x(t, \epsilon) \geq x_0 + \epsilon + y_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} [f(\tau, x(\tau, \epsilon)) + \epsilon]\tau d\tau.
\]

By theorem 1, we get \( x(t) < x(t, \epsilon) \) on \([0, \beta_0]\) for every \( \epsilon > 0 \). The uniqueness of the maximal solution shows that \( x(t, \epsilon) \) tends uniformly to \( \eta(t) \) as \( \epsilon \to 0 \). The proof is therefore complete.

### 4 The existence of global solutions

We need the following comparison result before we proceed further.

**Theorem 5** Assume that \( m \in C^1([0, T], R_+), g \in C([0, T], R_+), g(t, u) \) is nondecreasing in \( u \) for each \( t \) and

\[
m(t) \leq m(0) + m'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, m(\tau))d\tau
\]

where \( 0 \leq t \leq T \).

Let \( \eta(t) \) be the maximal solution of

\[
\left\{\begin{array}{l}
D^\alpha u = g(t, u), 0 \leq t \leq T \\
u(0) = u_0, u'(0) = u'_0 \geq 0
\end{array}\right.
\]

such that \( m(0) \leq u(0), m'(0) \leq u'(0) \), then we have

\[
m(t) \leq \eta(t), 0 \leq t \leq T.
\]
Theorem 6 Assume that \( f \in C([0, \infty) \times R), R \), \( g \in C([0, \infty) \times R_+, R_+), g(t, u) \) is nondecreasing in \( u \) for each \( t \) and
\[ |f(t, x)| \leq g(t, |x|) \]  \( \text{(21)} \)

Suppose further that we have local existence of solutions \( x(t, x_0, y_0) \) of
\[ \begin{cases} D^\alpha x(t) = f(t, x(t)) \\ x(0) = x_0, x'(0) = y_0 \end{cases} \]  \( \text{(22)} \)

and the maximal solution \( \eta(t) \) of
\[ \begin{cases} D^\alpha u = g(t, u), t \in [0, \infty) \\ u(0) = u_0 \geq 0, u'(0) = u'_0 \geq 0 \end{cases} \]

Then the largest integral of existence of an solution \( x(t, x_0, y_0) \) of (22) such that \( |x_0| \leq u_0, |y_0| \leq u'(0) \) is \( [0, \infty) \).

**proof.** Let \( x(t, x_0, y_0) \) be any solution of (22) such that \( |x_0| \leq u_0, |y_0| \leq u'(0) \), which exists on \([0, \beta)\) for \( 0 < \beta < \infty \) and the value of \( \beta \) cannot be increased further. Set \( m(t) = |x(t, x_0)| \) for \( 0 \leq t < \beta \), then using the assumption (21), we get
\[ m(t) \leq |x(t)| + |x'|t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} g(\tau, m(\tau)) d\tau. \]  \( \text{(20)} \)

Applying the comparison Theorem 5, we obtain
\[ m(t) = |x(t, x_0)| < \eta(t), 0 \leq t < \beta. \]

Since \( \eta(t) \) is assumed to exist on \([0, \infty)\). It follows that
\[ |g(t, \eta(t))| \leq M, 0 \leq t < \beta. \]

Now, let \( 0 \leq t_1 \leq t_2 < \beta \), then employing the arguments similar to estimate (13) and using (21) and the bound \( M \) of \( g \), we arrive at
\[ |x(t_1, x_0) - x(t_2, x_0)| \leq \left[ \frac{2M}{\Gamma(\alpha + 1)} + |y_0| \right](t_2 - t_1)^\alpha. \]

Letting \( t_1, t_2 \to \beta^- \) and using Cauchy criterion, it follows that \( \lim_{\beta \to \beta^-} x(t, x_0) \) exists. We define \( x(\beta, x_0) = \lim_{\beta \to \beta^-} x(t, x_0) \) and consider the new IVP
\[ \begin{cases} D^\alpha x = f(t, x(t)) \\ x(\beta) = x(\beta, x_0), x'(\beta) = x'(\beta, x_0) \end{cases} \]

By the assumed local existence, we find that \( x(t, x_0) \) can be continued beyond \( \beta \), contradicting our assumption. Hence every solution \( x(t, x_0) \) of (22) exists on \([0, \infty)\) and the proof is complete.

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