Auto-Bäcklund Transformation and Modified F-expansion Method to Find New Exact Solutions for the Variable Coefficients Generalized Zakharov-Kuznetsov Equation

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Abstract. In this paper, two methods are used to obtain new exact solutions for the generalized Zakharov-Kuznetsov equation with variable coefficients, the first one is auto-Bäcklund (BT) transformation method based on the homogeneous balance method and the second is the modified F-expansion function method. New exact solutions are obtained in the form of soliton-like solutions, Jacobi elliptic wave function-like solutions and trigonometric functions solutions.

Keywords: generalized Zakharov-Kuznetsov equation; auto-Bäcklund transformation; F-expansion function method; soliton-like solutions.

1 Introduction

Constructing exact solutions to partial differential equations (PDEs) is an important problem in mathematical physics. In order to obtain the exact solutions of NPDEs, various powerful methods have been presented, such as the inverse scattering method [1], Hirota method [2], symmetry method [3-5], homogeneous balance method [6-10], tanh method [11-12], Fan sub-equation method [13-15] and so on.

In this paper we will consider the following variable coefficients generalized Zakharov–Kuznetsov equation (VGZK)

\[ u_t + \alpha(t)u u_x + \beta(t)u^2 u_x + u_{xxx} + \gamma(t)u_{xyy} = 0, \]  

(1)

where \( \alpha(t), \beta(t) \) and \( \gamma(t) \) are arbitrary smooth functions of \( t \). Eq. (1) includes considerable interesting equations, such as KdV equation, mKdV equation, ZK equation and mZK equation. For the special case \( \alpha(t) = 1, \beta(t) = 0 \) and \( \gamma(t) = k \) see [16]. Moussa et al derived similarity reductions and some explicit solutions of Eq. (1) in [4]. Exact solutions with solitons and periodic structures of Eq. (1) with \( \alpha(t) = \alpha, \beta = 0, \gamma(t) = 1 \) were obtained by sine–cosine method in [17]. Recently, Z.I. Yan et al derive the symmetries and corresponding reductions of Eq. (1) [18] also in [19], S. A. El-Wakil et al obtained exact solution by using the Exp-function method.

In this work we would like to use the homogeneous balance method to derive a BT of Eq. (1), via the BT soliton-like solution is obtained, then we have applied the modified F-expansion method for Eq. (1) and new exact solutions in the form of Jacobi elliptic wave functions which degenerate into hyperbolic and trigonometric functions have been deduced.

2 Bäcklund transformation

According to the idea of improved HB [6–10], we seek for Bäcklund transformation of Eq. (1) in the form

\[ u(x,t) = f'(w_x) + u_0, \]  

(2)

where \( f = f(w), w = w(x,t) \) and \( u_0 = u_0(x,t) \).

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Substituting (2) in (1) yields

\[
\begin{align*}
\left[ u_x^4 + \gamma(t) w_x^2 w_y^2 \right] f''' + [6 w_x w_{xx} + 4 \gamma(t) w_x w_y w_{xy} + \gamma(t) w_y^2 w_{xx} + \gamma(t) w_x^2 w_{yy} ] f'' \\
+ [w_x w_t + \alpha(t) u_0 w_x + \beta(t) u_0^2 w_x^2 + 3 w_x w_{xx} + 4 w_x w_{xxx} + 2 \gamma(t) w_{xy} + 2 \gamma w_x w_{xy}] f' \\
+ \gamma(t) w_x w_{xy} + 2 \gamma(t) w_y w_{xy}] f'' + [w_{xt} + \alpha(t) u_0 w_x + \alpha(t) u_0 w_x + 2 \beta(t) u_0 u_x w_x + \beta(t) w_x w_{xx} + \gamma(t) w_y w_{xy}] f' \\
+ \beta(t) u_0^2 w_x + w_{xxx} + \gamma(t) w_x w_{xy}] f'' + [\beta(t) (2u_0 w_x w_{xx} + u_0 w_y^2) + \alpha(t) w_x w_{xx}] f'' + u_0 + \alpha(t) u_0 u_0 x + \gamma(t) u_0 w_y = 0.
\end{align*}
\]

(3)

If we assume that

\[ f = c \ln w, \]

where \( c \) is an arbitrary constant. We have the following equalities

\[ f'^2 f'' = \frac{c^2}{6} f'''', \quad f' f'' = -\frac{c}{2} f'''', \quad f'^3 = \frac{c^2}{2} f''''. \]

(5)

By using (5), Eq. (3) can be written as the sum of some terms with \( f', f'', f''' \), setting their coefficients to zero will lead to

\[ u_x^4 + \gamma(t) w_x^2 w_y^2 + \frac{c^2}{6} \beta(t) w_x^2 = 0, \]

\[ 6w_x^2 w_{xx} + 4 \gamma(t) w_x w_y w_{xy} + \gamma(t) w_y^2 w_{xx} + \gamma(t) w_x^2 w_{yy} + \frac{c^2}{2} \beta(t) w_x^3 w_{xx} - c \beta(t) u_0 w_3 - \frac{c}{2} \alpha(t) w_x^2 = 0, \]

\[ w_x w_t + \alpha(t) u_0 w_x + \beta(t) u_0^2 w_x^2 + 3 w_x w_{xx} + 4 w_x w_{xxx} + 2 \gamma(t) w_{xy} + 2 \gamma w_x w_{xy} \]

\[ + \gamma(t) w_x w_{xy} + 2 \gamma(t) w_y w_{xy} - c[\gamma(t) (2u_0 w_x w_{xx} + u_0 w_y^2) + \alpha(t) w_x w_{xx}] = 0 \]

\[ w_{xt} + \alpha(t) u_0 w_x + \alpha(t) u_0 w_x + 2 \beta(t) u_0 u_x w_x + \beta(t) w_x w_{xx} + \gamma(t) w_y w_{xy} = 0 \]

\[ u_0 + \alpha(t) u_0 u_0 x + \beta(t) u_0^2 u_0 x + u_0 w_x + \gamma(t) u_0 w_y = 0. \]

(6)

The next crucial step is the assumption that \( w(x, y, t) = 1 + \exp(\lambda(t) \pm k_1 x \pm k_2 y) \) and \( u_0(x, y, t) = u_0 \) is arbitrary constant. Substituting into system of Eqs. (6), results in

\[ k_1^2 + \gamma(t) k_2^2 + \frac{c^2}{6} \beta(t) k_1^2 = 0, \]

\[ 6k_1^2 \pm 6 \gamma(t) k_2^2 + \frac{c^2}{2} \beta(t) k_2^2 + (\beta(t) u_0 + \frac{1}{2} \alpha(t)) c k_1 = 0, \]

\[ \pm \lambda'(t) + u_0 k_1 (\alpha(t) + u_0 \beta(t)) + 7k_1^2 + 7 \gamma(t) k_1 k_2^2 = c k_1^2 [2u_0 \beta(t) + \alpha(t)] = 0, \]

\[ \pm \lambda'(t) + (\alpha(t) + u_0 \beta(t)) u_0 k_1 + k_1^2 + \gamma(t) k_1 k_2^2 = 0. \]

(7)

By solving the above system, we have

\[ \gamma(t) = -\frac{k_1^2}{k_2^2}(1 + \frac{c^2}{6} \beta(t)), \quad k_2 \neq 0 \]

\[ \alpha(t) = -(2u_0 \mp c k_1) \beta(t), \]

\[ \lambda(t) = [\pm \frac{1}{6} k_1^2 c^2 + k_1^2 u_0 \pm k_1 u_0^2] \int \beta(t) dt + c_0, \]

(8)

where \( c_0 \) is an integration constant. Substitute from Eqs. (8) and (4) in Eq. (2), we get the following exact soliton solution for the VGZK equation

\[ u(x, y, t) = \pm \frac{ck_1}{2} \left[ 1 - \tanh \left[ \frac{1}{2} \left( \pm \frac{1}{6} k_1^2 c^2 + k_1^2 u_0 \pm k_1 u_0^2 \right) \int \beta(t) dt \pm k_1 x \pm k_2 y + c_0 \right] \right] + u_0 \]

(9)
3 Modified F-expansion method

In order to obtain more periodic wave solutions expressed by various Jacobi elliptic functions for the GVZK equation, we applied the modified F-expansion method [20]. We briefly describe this method as follows:

Step (1). For a given NPDEs with independent variables \(X = (x, y, t)\) and dependent variable \(u\):

\[
P(u, u_t, u_x, \ldots, u_{xx}, u_{yy}, \ldots) = 0,
\]

where \(P\) is a polynomial, \(u\) is an arbitrary function of \(t\), and hence holds for \(A\) where \(A\) is an arbitrary function of \(t\), both to be determined later. Substituting \((11)\) in \((10)\) yields an ordinary differential equation (ode) for \(f(\xi)\)

\[
\ddot{F}(f, f', k_1 f', k_1^2 f'', k_2^2 f'', \ldots) = 0
\]

Step (2). Suppose that \(f(\xi)\) can be expressed by a finite power series of \(F(\xi)\)

\[
f(\xi) = \sum_{k=0}^{n} A_k F^k(\xi)
\]

where \(A_k, k = 1, 2, \ldots, n\) are arbitrary constants and \(F(\xi)\) satisfies the first order nonlinear ode

\[
F' = q_0 + q_2 F^2(\xi) + q_4 F^4(\xi),
\]

and hence holds for \(F(\xi)\)

\[
\begin{cases}
F'' = q_2 F + 2q_4 F^3, \\
F''' = (q_2 + 6q_4 F^2)F'.
\end{cases}
\]

Eq. (14) have many solutions (see the appendix).

Step (3). To determine the integer \(n\) in Eq. (13), balance the nonlinear term with the highest order derivatives of \(f(\xi)\) in Eq. (12), then substitute from Eq. (13) into Eq. (12) and use Eqs. (14) and (15), then Eq. (12) converted into a polynomial in \(F(\xi)\). Setting each coefficient of the polynomial to zero yields equations for \(A_1, A_2, A_3, \ldots, c_1, c_2, \ldots, q_0, q_2, q_4\).

Step (4). Solve the equations obtained in step 3, with the aid of Mathematica substitute these results into Eq. (13) using the values of \(F\) given in the appendix, then many exact traveling wave solutions of Eq. (10) will be obtained.

3.1 Application to the variable coefficients generalized Zakharov-Kuznetsov equation

Now we going to apply the modified F-expansion method to the VGZK equation as follows:

Step (1). Suppose the solution of Eq. (1) is of the form

\[
u = f(\xi), \quad \xi = c_1 x + c_2 y + \int \mu(t) dt
\]

Then Eq. (1) become

\[
\mu(t) f' + c_1 \alpha(t) f f' + \beta(t) c_1 f^2 f' + c_1^2 f''' + \gamma(t) c_1^2 c_2 f'' = 0,
\]

Integrate the above Eq. with respect to \(\xi\), yields

\[
\mu(t) f + \frac{c_1}{2} \alpha(t) f^2 + \frac{c_1}{3} \beta(t) f^3 + c_1 [c_1^2 + \gamma(t) c_2^2] f'' = 0,
\]

Step (2). Substituting (13) into Eq. (18), considering the homogeneous balance between \(f^3\) and \(f''\) yields that \(n = 1\), suppose the solution of ode (18) is of the form

\[
f(\xi) = A_0 + A_1 F(\xi)
\]
Step (3). Substituting (19) and (14) into Eq. (18) using relations (15), yields
\[
\frac{1}{3} A_1^3 c_1 \beta (t) + 2 A_1 q_4 \left( c_1^2 + c_1 c_2^2 (t) \right) \] F^3 + \left[ \frac{1}{2} A_1^2 c_1 \alpha (t) + A_0 A_1^2 c_1 \beta (t) \right] F^2
\[
+ A_0 A_1 c_1 \alpha (t) + A_0^2 A_1 c_1 \beta (t) + A_1 q_2 \left( c_1^2 + c_1 c_2^2 \gamma (t) \right) + A_1 \mu (t) \] F
\[
+ \frac{1}{2} c_1 A_0^2 c_1 \alpha (t) + \frac{1}{3} A_0^3 c_1 \beta (t) + A_0 \mu (t) = 0 \quad (20)
\]
Setting each coefficient of \( F^k \) \((k = 0, 1, 2, 3)\) to zero yields a set of equations for \( A_0, A_1, c_1, c_2, q_0, q_2, q_4, \alpha (t), \beta (t), \gamma (t) \) and \( \mu (t) \)
\[
\frac{1}{2} c_1 A_0^2 \alpha (t) + \frac{1}{3} A_0^3 c_1 \beta (t) + A_0 \mu (t) = 0,
\]
\[
A_0 A_1 c_1 \alpha (t) + A_0^2 A_1 c_1 \beta (t) + A_1 q_2 \left( c_1^2 + c_1 c_2^2 \gamma (t) \right) + A_1 \mu (t) = 0,
\]
\[
\frac{1}{2} A_1^2 c_1 \alpha (t) + A_0 A_1^2 c_1 \beta (t) = 0,
\]
\[
\frac{1}{3} A_1^3 c_1 \beta (t) + 2 A_1 q_4 \left( c_1^2 + c_1 c_2^2 \gamma (t) \right) = 0 \quad (21)
\]
Step (4). The solution of the above system is
\[
\mu (t) = \frac{2}{3} c_1 A_0^2 \beta (t),
\]
\[
\alpha (t) = -2 A_0 \beta (t), \quad \gamma (t) = \frac{-3 c_1^2 q_2 + A_0^2 \beta (t)}{3 c_1^2 q_2},
\]
\[
A_1 = \pm i A_0 \sqrt{\frac{2 q_4}{q_2}} \quad (24)
\]
Substituting from (24) into (19) and from (22) into (16) yield the general form solution to the VGZK equation
\[
u (x, y, t) = A_0 e^{i A_0 \sqrt{\frac{2 q_4}{q_2}}} F (\xi),
\]
with
\[
\xi = c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \quad (26)
\]
Step (5). Using the values of \( F \) given in table (1) in the appendix, we get the following exact solutions for the VGZK equation in the form of Jacobi elliptic functions
\[
u (x, y, t) = A_0 \left[ 1 - \sqrt{\frac{2 m^2}{1 + m^2}} \text{sn} \left( c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \right) \right] \quad (27)
\]
\[
u (x, y, t) = A_0 \left[ 1 - \sqrt{\frac{2 m^2}{1 + m^2}} \text{cd} \left( c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \right) \right] \quad (28)
\]
\[
u (x, y, t) = A_0 \left[ 1 - \sqrt{\frac{2 m^2}{2 m^2 - 1}} \text{cn} \left( c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \right) \right] \quad (29)
\]
\[
u (x, y, t) = A_0 \left[ 1 + \sqrt{\frac{2}{2 - m^2}} \text{dn} \left( c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \right) \right] \quad (30)
\]
\[
u (x, y, t) = A_0 \left[ 1 - \sqrt{\frac{2}{1 + m^2}} \text{ns} \left( c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \right) \right] \quad (31)
\]
\[
u (x, y, t) = A_0 \left[ 1 - \sqrt{\frac{2}{1 + m^2}} \text{dc} \left( c_1 x + c_2 y + \frac{2}{3} c_1 A_0^2 \int \beta (t) \, dt \right) \right] \quad (32)
\]
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and from the above solutions we get the following new exact solution by Substituting in Eqs. (27), others give trivial solutions (42) noting that other Eqs. give constant and imaginary solutions.

Also we have applied the modified F-expansion to this equation and new exact solutions in the form of Jacobi elliptic wave functions which degenerate into hyperbolic and trigonometric functions.

4 Conclusion

In this paper we have applied BT method to the VGZK equation and new exact solitary wave solution has been obtained also we have applied the modified F-expansion to this equation and new exact solutions in the form of Jacobi elliptic wave functions which degenerate into hyperbolic and trigonometric functions.

All solutions obtained in this paper have been checked by Mathematica software.

Appendix

Relations between values of \((q_0, q_2, q_4)\) and corresponding \(F(w)\) in the ODE (14)

<table>
<thead>
<tr>
<th>(q_0)</th>
<th>(q_2)</th>
<th>(q_4)</th>
<th>(F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(- (1 + m^2))</td>
<td>(m^2)</td>
<td>(sn(w), cd(w))</td>
</tr>
<tr>
<td>(1 - m^2)</td>
<td>(2m^2 - 1)</td>
<td>(-m^2)</td>
<td>(cn(w))</td>
</tr>
<tr>
<td>(m^2 - 1)</td>
<td>(2 - m^2)</td>
<td>(-1)</td>
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</tr>
<tr>
<td>1</td>
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<td>(-m^2 (1 - m^2))</td>
<td>(sd(w))</td>
</tr>
<tr>
<td>(1 - m^2)</td>
<td>(2 - m^2)</td>
<td>1</td>
<td>(cs(w))</td>
</tr>
<tr>
<td>(-m^2 (1 - m^2))</td>
<td>(2m^2 - 1)</td>
<td>1</td>
<td>(ds(w))</td>
</tr>
</tbody>
</table>

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where $m$ is the modulus of the Jacobi elliptic functions which satisfies $0 \leq m \leq 1$.

The Jacobi elliptic functions degenerate as hyperbolic functions when $m \to 1$, and

\[
\begin{array}{cccccccc}
\text{sn}(w) & \text{cn}(w) & \text{dn}(w) & \text{se}(w) & \text{sd}(w) & \text{cd}(w) \\
\text{tanh}(w) & \text{sech}(w) & \text{sech}(w) & \text{sinh}(w) & \text{sinh}(w) & 1 \\
\text{cs}(w) & \text{cosh}(w) & \text{cosh}(w) & \text{csch}(w) & \text{csch}(w) & 1 \\
\end{array}
\]

The Jacobi elliptic functions degenerate as trigonometric functions when $m \to 0$,

\[
\begin{array}{cccccccc}
\text{sn}(w) & \text{cn}(w) & \text{dn}(w) & \text{se}(w) & \text{sd}(w) & \text{cd}(w) \\
\sin(w) & \cos(w) & 1 & \tan(w) & \sin(w) & \cos(w) \\
\text{cs}(w) & \text{sc}(w) & \text{cd}(w) & \text{ds}(w) & \text{dc}(w) & 1 \\
\end{array}
\]

References


