The First Integral Method for Huxley Equation

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Abstract. The first integral method is used to obtain the exact travelling wave solutions of the Huxley equation. It is shown that the method is effective and direct method, based on the ring theory of commutative algebra.

Keywords: Travelling wave, First integral method, Huxley equation

1 Introduction

The study of solitary wave solutions of nonlinear partial differential equations (NLPDEs) in mathematical physics has enjoyed an intense period of activity over the past decades. In recent years, the direct search for exact solutions of PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform the complicated and tedious algebraic calculations on computer.

In recent years, many ansatz methods have been developed, such as, tanh method [1,2], extended tanh function method [3], modified extended tanh function method [4,5], sine-cosine method [6], tanh-coth method [7], exp-function method [8-10], homogeneous balance method [11], etc. Feng [12], proposed a new powerful method, the first integral method, based on the ring theory of commutative algebra. The first integral method has been used to study the travelling wave solutions of various nonlinear evolution equations [13-15].

In this paper, we extend the application of first integral method to solve the well known Huxley equation. In section 2, we proposed the analysis of the method for finding exact travelling wave solutions of NLPDEs. In section 3, we established the exact travelling wave solution for Huxley equation. Finally, in section 4, conclusions of the analysis are given.

2 Basic idea of The First Integral Method

Consider the nonlinear partial differential equation of the form:

\[ F(u, u_t, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0 \]  \hspace{1cm} (1)

where \( u(x, t) \) is the solution of nonlinear partial differential equation (1). The transformations

\[ u(x, t) = f(\xi), \text{ where } \xi = x - ct \]  \hspace{1cm} (2)

enables us to use the following changes:

\[ \frac{\partial}{\partial t}() = -c \frac{d}{d\xi}(); \quad \frac{\partial}{\partial x}() = \frac{d}{d\xi}(); \quad \frac{\partial^2}{\partial x^2}() = \frac{d^2}{d\xi^2}(), \ldots \]  \hspace{1cm} (3)

Using (3), we can transform equation (1) to the following equation:

\[ G(f, f_{\xi}, f_{\xi\xi}, \ldots) = 0 \]  \hspace{1cm} (4)

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Introducing new independent variables
\[ X(\xi) = f(\xi), \quad Y(\xi) = f_\xi(\xi) \] (5)
we obtain the following system of ordinary differential equations (ODEs):
\[
\begin{cases}
X_\xi(\xi) = Y(\xi); \\
Y_\xi(\xi) = F(X(\xi), Y(\xi))
\end{cases}
\] (6)

From the qualitative theory of ODE’s [16], if one can find the first integrals of Eq. (6) then the general solutions of Eq. (6) can be found directly. However, in general, it rarely happens, because for a given plane autonomous system, there is no systematic theory that tells us how to find its first integrals, also there is no logical way which tells what these first integrals are. So, we will apply the Division Theorem to obtain the first integral of equations (6) under various parameter conditions, which reduces (4) to a first order integrable ODE. An exact solution of (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

**Division Theorem:** Suppose that \( P(w, z) \) and \( Q(w, z) \) are polynomials of two variables \( w \) and \( z \) in domain \( \mathbb{C}[w, z] \) and \( P(w, z) \) is irreducible in domain \( \mathbb{C}[w, z] \). If \( Q(w, z) \) vanishes at all zero points of \( P(w, z) \), then there exists a polynomial \( G(w, z) \) in domain \( \mathbb{C}[w, z] \) such that
\[ Q(w, z) = P(w, z)G(w, z). \]

Feng [17], pointed out, that the Division Theorem follows immediately from Hilbert-Nullstellensatz Theorem [18] of commutative algebra.

### 3 Application to Huxley Equation

Huxley equation is an evolution equation that describes the nerve propagation in biology. It also gives a phenomenological description of the behavior of the myosin heads II. This equation has many fascinating phenomena such as bursting oscillation [19], interspike [20], bifurcation, and chaos [21]. We apply the above method to nonlinear Huxley equation [22-26] of the form
\[ u_t = u_{xx} + u(k - u)(u - 1). \] (7)

Using (2) and (3), in eq. (7) we receive
\[ -cf' = f'' + f(k - f)(f - 1) \] (8)

Use of Equation (5) in (8) leads to the following system of ODEs:
\[
\begin{align*}
\dot{X}(\xi) &= Y(\xi) \\
\dot{Y}(\xi) &= -cY(\xi) - X(\xi)(k - X(\xi))(X(\xi) - 1)
\end{align*}
\] (9)

According to the first integral method, suppose that \( X(\xi) \) and \( Y(\xi) \) are the nontrivial solutions of (9), and
\[ q(X, Y) = \sum a_i(X)Y^i = 0 \] (10)
is an irreducible polynomial in the complex domain \( \mathbb{C}[X, Y] \) such that
\[ q[X(\xi), Y(\xi)] = \sum_{i=0}^m a_i(X)Y^i = 0 \] (10)
where \( a_i(X) \), \( (i = 0, 1, ..., m) \) are polynomials of \( X \) and \( a_m(X) \neq 0 \). Equation (10) is called the first integral of (9), due to the Division Theorem, there exists a polynomial \( g(X) + h(X)Y \) in the complex domain \( \mathbb{C}[X, Y] \) such that
\[
\frac{dq}{d\xi} = \frac{\partial q}{\partial X} \frac{\partial X}{d\xi} + \frac{\partial q}{\partial Y} \frac{\partial Y}{d\xi} = (g(X) + h(X)Y) \sum_{i=0}^m a_i(X)Y^i \] (11)

Similarly, for the case of (14b)-(14f), the exact travelling wave solutions are written as combining (15) with (9), we obtain the exact solution of (8) and then the exact solution of Huxley equation (7) can be obtained.

Substituting \( a_0(X), a_1(X) \) and \( g(X) \) in equation (12c) and setting all the coefficients of powers of \( X \) equal to zero, then we obtain a system of nonlinear algebraic equations and using Maple solving them, we obtain

\[
\begin{align*}
A_0 &= 0, A_1 = \sqrt{2}, c = -\frac{1}{\sqrt{2}} (k + 1), B_0 = \frac{1}{\sqrt{2}} k \\
A_0 &= 0, A_1 = -\sqrt{2}, c = \frac{1}{\sqrt{2}} (k + 1), B_0 = -\frac{1}{\sqrt{2}} k \\
A_0 &= -\sqrt{2}, A_1 = \sqrt{2}, c = -\frac{1}{\sqrt{2}} (k - 2), B_0 = 0 \\
A_0 &= \sqrt{2}, A_1 = -\sqrt{2}, c = \frac{1}{\sqrt{2}} (k - 2), B_0 = 0 \\
A_0 &= -\sqrt{2}k, A_1 = \sqrt{2}, c = \frac{1}{\sqrt{2}} (2k - 1), B_0 = 0 \\
A_0 &= \sqrt{2}k, A_1 = -\sqrt{2}, c = -\frac{1}{\sqrt{2}} (2k - 1), B_0 = 0
\end{align*}
\]

Using (14a) into (13) and (10), we obtain

\[
Y = -\frac{X^2 + (k + 1)X - k}{\sqrt{2}}
\]

Combining (15) with (9), we obtain the exact solution of (8) and then the exact solution of Huxley equation (7) can be written as

\[
u_1(x,t) = \frac{1 + ke^{\left(\frac{\sqrt{2}}{k+1}t\right)}(k-1)}{1 + e^{\left(\frac{\sqrt{2}}{k+1}t\right)}(k-1)}
\]

where \( \alpha \) is an arbitrary constant. Thus the travelling wave solution of Huxley Equation (7) can be written as

\[
u_1(x,t) = \frac{-1 + ke^{\left(\frac{\sqrt{2}}{k+1}t\right)}(k-1)}{-1 + e^{\left(\frac{\sqrt{2}}{k+1}t\right)}(k-1)}
\]

Similarly, for the case of (14b)-(14f), the exact travelling wave solutions are

\[
u_2(x,t) = \frac{-k + e^{\left(\frac{\sqrt{2}}{k+1}t\right)}(k-1)}{-1 + e^{\left(\frac{\sqrt{2}}{k+1}t\right)}(k-1)}
\]

\[
u_3(x,t) = \sqrt{kx} \tanh \left[ \left( \frac{1}{\sqrt{2}} \left( x + \frac{1}{\sqrt{2}} (k - 2) t \right) + \alpha \right) \sqrt{kx} \right]
\]
The solutions are new exact solutions.

Using (22a) into (20), (21) and (10), we obtain

\begin{align}
    u_4(x, t) &= -\sqrt{kx} \tanh \left( \left( \frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}} (k - 2)t \right) + \alpha \right) \sqrt{kx} \\
    u_5(x, t) &= \frac{1}{-1 + e^{-\frac{1}{\sqrt{2}} (x - \frac{1}{\sqrt{2}} (2k - 1)t) + \alpha}} \\
    u_6(x, t) &= \frac{-1}{-1 + e^{-\frac{1}{\sqrt{2}} (x + \frac{1}{\sqrt{2}} (2k - 1)t) + \alpha}}
\end{align}

The solutions are new exact solutions.

**Case II:** Suppose the \( m = 2 \), by equating the coefficients of \( Y^i(i = 0, 1, 2, 3) \) on both sides of equation (11), we have

\begin{align}
    \dot{a}_2(X) &= h(X)a_2(X) \\
    \dot{a}_1(X) &= (2c + g(X))a_2(X) + a_1(X)h(X) \\
    \dot{a}_0(X) &= (c + g(X))a_1(X) + 2a_2(X)(k - X)(X - 1) + h(X)a_0(X) \\
    -a_1(X)X(k - X)(X - 1) &= g(X)a_0(X)
\end{align}

Since \( a_i(X)(i = 0, 1, 2) \) are polynomials, then from (18a) we deduce that \( a_2(X) \) is constant and \( h(X) = 0 \). For simplicity, let us take \( a_2(X) = 1 \), so (18a)-(18d) can be written as

\begin{align}
    \dot{a}_2(X) &= 1 \\
    \dot{a}_1(X) &= (2c + g(X)) \\
    \dot{a}_0(X) &= (c + g(X))a_1(X) + 2X(k - X)(X - 1) + h(X)a_0(X) \\
    -a_1(X)(k - X)(X - 1) &= g(X)a_0(X)
\end{align}

Balancing the degrees of \( g(X), a_1(X) \) and \( a_0(X) \), we conclude that either \( \deg(g(X)) = 0 \) or \( \deg(g(X)) = 1 \).

**Case II (i):** Taking \( \deg(g(X)) = 0 \), and suppose that \( g(X) = A_1 \), then we obtain \( a_1(X) \) and \( a_0(X) \) as

\begin{align}
    a_1(X) &= (2c + A_1)X + A_0 \\
    a_0(X) &= d + (cA_0 + A_1A_0)X + \\
    &\frac{1}{2} (2c^2 + 3cA_1 + A_1^2 - 2k)X^2 + \frac{1}{3} (2k^2 + 2k + 2)X^3 - \frac{1}{2} X^4
\end{align}

where \( A_0 \) and \( d \) are constants. Substituting \( g(X), a_0(X), a_1(X) \) and \( a_2(X) \) in (19d) and setting all the coefficients of powers of \( X \) equal to zero, then we obtain a system of nonlinear algebraic equations and using Maple solving them, we obtain

\begin{align}
    d &= 0, A_0 = 0, A_1 = 2\sqrt{2}, k = -1, c = -\frac{3}{2}\sqrt{2} \\
    d &= 0, A_0 = 0, A_1 = -2\sqrt{2}, k = -1, c = \frac{3}{2}\sqrt{2} \\
    d &= 0, A_0 = \frac{\sqrt{2}}{4}, A_1 = \sqrt{2}, k = \frac{1}{2}, c = -\frac{3}{4}\sqrt{2} \\
    d &= 0, A_0 = -\frac{\sqrt{2}}{4}, A_1 = -\sqrt{2}, k = \frac{1}{2}, c = \frac{3}{4}\sqrt{2} \\
    d &= 0, A_0 = 2\sqrt{2}, A_1 = 2\sqrt{2}, k = -1, c = -\frac{3}{2}\sqrt{2} \\
    d &= 0, A_0 = -2\sqrt{2}, A_1 = -2\sqrt{2}, k = 2, c = \frac{3}{2}\sqrt{2}
\end{align}

Using (22a) into (20), (21) and (10), we obtain

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Similarly, for the case of (22b)-(22f), the exact travelling wave solutions are combining (23) with (9), we obtain the exact solution of (8) and then the exact solution of Huxley equation (7) can be written as

$$u_7(x, t) = \frac{e^{\frac{\sqrt{2}}{2}(x + \frac{\sqrt{2}}{4}t)^\alpha}}{1 + e^{\frac{\sqrt{2}}{2}(x + \frac{\sqrt{2}}{4}t)^\alpha}}, \quad u_8(x, t) = \frac{e^{\frac{\sqrt{2}}{2}(x + \frac{\sqrt{2}}{4}t)^\alpha}}{1 + e^{\frac{\sqrt{2}}{2}(x + \frac{\sqrt{2}}{4}t)^\alpha}}$$

(24a)

Similarly, for the case of (22b)-(22f), the exact travelling wave solutions are

$$u_9(x, t) = \frac{1}{1 + e^{\frac{\sqrt{2}}{2}(x - \frac{\sqrt{2}}{4}t)^\alpha}}, \quad u_{10}(x, t) = \frac{1}{1 + e^{\frac{\sqrt{2}}{2}(x - \frac{\sqrt{2}}{4}t)^\alpha}}$$

(24b)

$$u_{11}(x, t) = \frac{1}{-2 + e^{\frac{\sqrt{2}}{4}(x + \frac{\sqrt{2}}{4}t)^2 + 2\alpha}}, \quad u_{12}(x, t) = \frac{1}{-2 + e^{\frac{\sqrt{2}}{4}(x + \frac{\sqrt{2}}{4}t)^2 + 2\alpha}}$$

(24c)

$$u_{13}(x, t) = \frac{1}{-1 + 2e^{\frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{4}t)^2 + 2\alpha}}, \quad u_{14}(x, t) = \frac{1}{-1 + 2e^{\frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{4}t)^2 + 2\alpha}}$$

(24d)

$$u_{15}(x, t) = \frac{1}{-1 + e^{\frac{\sqrt{2}}{4}(x + \frac{\sqrt{2}}{4}t)^\alpha}}, \quad u_{16}(x, t) = \frac{1}{-1 + e^{\frac{\sqrt{2}}{4}(x + \frac{\sqrt{2}}{4}t)^\alpha}}$$

(24e)

$$u_{17}(x, t) = \frac{-2 + e^{\frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{4}t)^\alpha}}{-1 + e^{\frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{4}t)^\alpha}}, \quad u_{18}(x, t) = \frac{-2 + e^{\frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{4}t)^\alpha}}{-1 + e^{\frac{\sqrt{2}}{4}(x - \frac{\sqrt{2}}{4}t)^\alpha}}$$

(24f)

These are all new exact solutions.

**Case II (ii):** Taking \(\text{deg}(g(X)) = 1\), and suppose that \(g(X) = A_1 X + A_0\) and \(A_1 \neq 0\), then we obtain \(a_1(X)\) and \(a_0(X)\) as

$$a_1(X) = B_0 + (2c + A_0)X + \frac{1}{2}A_1X^2$$

(25)

$$a_0(X) = d + (cB_0 + A_0B_0)X + \frac{1}{2}(2c^2 + 3cA_0 + A_0^2 + A_1B_0 - 2k)X^2 + \frac{1}{6}(5cA_1 + 3A_0A_1 + 4k + 4)X^3 + \frac{1}{8}(A_1^2 - 4)X^4$$

(26)

where \(B_0\) and \(d\) are constants. Substituting \(g(X)\), \(a_0(X)\), \(a_1(X)\) and \(a_2(X)\) in (19d) and setting all the coefficients of powers of \(X\) equal to zero, then we obtain a system of nonlinear algebraic equations and using Maple solving them, we obtain

$$d = \frac{1}{2}, \quad c = 0, \quad A_0 = 0, \quad A_1 = -2\sqrt{2}, \quad k = -1, \quad B_0 = \sqrt{2}$$

(27a)

$$d = \frac{1}{2}, \quad c = 0, \quad A_0 = 0, \quad A_1 = 2\sqrt{2}, \quad k = -1, \quad B_0 = -\sqrt{2}$$

(27b)

$$d = 0, \quad c = -\frac{\sqrt{2}}{2}(k - 2), \quad A_0 = -2\sqrt{2}, \quad A_1 = 2\sqrt{2}, \quad B_0 = 0, \quad k = k$$

(27c)

$$d = 0, \quad c = \frac{\sqrt{2}}{2}(k - 2), \quad A_0 = 2\sqrt{2}, \quad A_1 = -2\sqrt{2}, \quad B_0 = 0, \quad k = k$$

(27d)

$$d = 0, \quad c = \frac{\sqrt{2}}{2}(2k - 1), \quad A_0 = -2\sqrt{2}k, \quad A_1 = 2\sqrt{2}k, \quad B_0 = 0, \quad k = k$$

(27e)

$$d = 0, \quad c = -\frac{\sqrt{2}}{2}(2k - 1), \quad A_0 = 2\sqrt{2}k, \quad A_1 = -2\sqrt{2}, \quad B_0 = 0, \quad k = k$$

(27f)
\[ d = \frac{1}{2} k^2, c = -\frac{\sqrt{2}}{2}(k + 1), A_0 = 0, A_1 = 2\sqrt{2}, B_0 = \sqrt{2}k, k = k \]

\[ d = \frac{1}{2} k^2, c = \frac{\sqrt{2}}{2}(k + 1), A_0 = 0, A_1 = -2\sqrt{2}, B_0 = -\sqrt{2}k, k = k \]

Similarly, for the cases of (27a)-(27b), the exact solutions of the Huxley equation (7) are

\[ u_{19} = -\tanh \left( \frac{1}{\sqrt{2}} x + \alpha \right) \]  

\[ u_{20} = \tanh \left( \frac{1}{\sqrt{2}} x + \alpha \right) \]

for the cases (27c)-(27h) the solutions are same as previously obtained solutions.

A comparison between our results and the results of Zhou [27] obtained by Exp-function Method, shows that some of the obtained results are new and the rest are the same. We emphasise that our results can be found to have potentially useful applications in mathematical physics and applied mathematics including numerical simulation.

4 Conclusion

The first integral method was successfully applied to obtain the exact travelling wave solutions of the Huxley equation. The method is easier and quicker than other traditional methods. Also, it is direct, concise and more specifically computerizable. The symbolic manipulation software Maple was used to solve complicated and tedious algebraic calculations. Therefore, the proposed method can be extended to solve the nonlinear problems which arise in the soliton theory and other areas.

References


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