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Global Conservative Solutions of the Two-Component Camassa-Holm Shallow Water System

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Abstract: In this paper, we show that the two-component Camassa-Holm equation possesses a global continuous semigroup of weak conservative solutions for initial data. The result is obtained by introducing a new set of independent and dependent variables, this system is transformed into a semilinear system, whose solutions are obtained as fixed points of a contractive transformation. To characterize conservative solutions it is necessary to show that the total energy equals a constant, for almost every time.

Keywords: two-component Camassa-Holm system; global solutions ; conservative solutions

1 Introduction

We consider in this paper the following integrable two-component Camassa-Holm shallow water system [1]:

$$\begin{cases} m_t - Au + 2u_x m + u m_x + \rho\rho_x = 0, & t > 0,\ x \in R, \\ m = u - u_{xx}, & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in R. \end{cases} \tag{1}$$

where the variable $u(t,x)$ describes the horizontal velocity of the fluid and the variable $\rho(t,x)$ is in connection with the free surface elevation from equilibrium (or scalar density) with the boundary assumptions, $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The system (1) is integrable [1] in the sense that it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ζ :$\Psi_{xx} = \left(-\zeta^2\rho^2 + \zeta\left(m - \frac{A}{2}\right) + \frac{1}{4}\right)\Psi$,$\Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi$. It has a bi-Hamiltonian structure corresponding to the Hamiltonian$H_1 = \frac{1}{2}\int\left(um + (\rho-1)^2\right)dx$and the Hamiltonian$H_2 = \frac{1}{2}\int\left(u(\rho-1)^2 + 2u(\rho-1) + u^3 + uu_x^2 - Au^2\right)dx$. There are two Casimirs: $\int(\rho-1)\,dx$ and$\int m dx$.

The system (1) with nonzero constant vorticity was rigorously justified by Ivanov [2] to approximate the governing equations for shallow water waves and the system (1) without vorticity was also rigorously justified by Constantin and Ivanov [1]. A modified two-component Camassa-Holm system which possesses singular solutions in componentρ was proposed recently by Holm, Nraigh and Tronci in [3]. The mathematical properties of (1) have been also studied further in many works. For example, local well-posedness for system (1) with initial data $(u_0,\rho_0) \in H^s \times H^{s-1}$ with $s \geq 2$ and some precise blow-up scenarios for strong solutions were investigated in [4]. Gui and Liu [5] recently obtained results of local well-posedness in the Besov spaces. Guan and Yin [6] studied global existence and blow-up phenomena for the system (1) with initial data $(u_0,\rho_0) \in H^s \times H^{s-1}$with $s > \frac{5}{2}$.

For $\rho \equiv 0$, Eq. (1) becomes the classical Camassa-Holm equation. It is known that the Camassa-Holm equation possesses a global continuous semigroup of weak conservative solutions for initial data$u_0 \in H^1$ [7]. The goal of this paper is to investigate whether or not the system (1) with $\sigma = 1$ has global conservative solutions as the classical Camassa-Holm equation in Sobolev space$H^s \times H^{s-1}$ for$s > \frac{3}{2}$. We first introduce a new set of independent and dependent variables in connection with smooth solutions. In these new variables, the evolution equation becomes semilinear. We can obtain a local solution as the fixed point of a contractive transformation. Solutions of the equivalent semilinear system can be globally extended in time, even after wave breaking. As their total energy is a.e. equal to a constant with respect to time, the solutions that we obtain are conservative. These will be analyzed in a forthcoming paper.

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2 An equivalent semilinear system to the two-component Camassa-Holm equation

The two-component Camassa-Holm shallow water system can be recast in the nonlocal conservation law:

$$\begin{cases} u_t + uu_x + \partial_x \left(1-\partial_x^2\right)^{-1} \left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2} - Au\right) = 0 \\ \rho_t + (u\rho)_x = 0 \end{cases} \tag{2}$$

with $m = u - u_{xx}$. Notice that if$p(x) := \frac{1}{2}e^{-|x|}$,$x \in R$, then$p * f = \left(1-\partial^2\right)^{-1} f$ for all $f \in L^2$. Using this identity, Eq. (2) takes the form of a quasi-linear evolution equation:

$$\begin{cases} u_t + uu_x + \partial_x p * \left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2} - Au\right) = 0 \\ \rho_t + (u\rho)_x = 0 \end{cases} \tag{3}$$

We consider the following form of the two-component Camassa-Holm shallow water system in this paper:

$$\begin{cases} u_t + uu_x + P_x = 0 \\ \rho_t + (u\rho)_x = 0 \end{cases} \tag{4}$$

with initial data

$$\begin{cases} u(0,x) = \bar{u}(x), & x \in R, \\ \rho(0,x) = \bar{\rho}(x), & x \in R. \end{cases} \tag{5}$$

Here the source term P is defined as$P = p * \left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2} - Au\right) = \left(1-\partial^2\right)^{-1} \left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2} - Au\right) \in H^1(R)$. To make sense of the source term P, at each time t we require that the function u in the space $H^1(R)$ with derivative $u_x \in L^2(R)$ and ρ in the space $L^2(R)$ are absolutely continuous, endowed with the norm$\|u\|_{H^1} = \left(\int_R \left[u^2 + u_x^2\right] dx\right)^{1/2}$.

Definition 1 *By a solution $z = (u\ ,\ \rho)$ of the Cauchy problem (4)-(5) on $[t_1\ , t_2]$, we define function $u = u(t,x)$ on $[t_1\ , t_2] \times R$ is H?lder continuous with the following properties. At each fixed t we have $u(t,\cdot) \in H^1(R)$. Moreover, the map $t \to u(t,\cdot)$ is Lipschitz continuous from $[t_1\ , t_2]$ into $L^2(R)$, satisfying the initial condition (5).*

Differentiating the first equation in (4) with respect to x and using the identity $f = u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2} - Au$, we have

$$u_{xt} + uu_{xx} + u_x^2 + P - \left(u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2} - Au\right) = 0. \tag{6}$$

Multiplying the first equation in (4) by u and the second by ρ , and multiplying (6) by u_x, we obtain the following conservation laws with source term

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + uP\right)_x - u_x P = 0, \tag{7}$$

$$\left(\frac{u_x^2}{2}\right)_t + \left(\frac{uu_x^2}{2} - \frac{u^3}{3} + \frac{A}{2}u^2\right)_x - \frac{u_x\rho^2}{2} + u_x P = 0 \tag{8}$$

$$\left(\frac{\rho^2}{2}\right)_t + u_x\rho^2 + u\rho\rho_x = 0. \tag{9}$$

valid for smooth solutions. Then, by adding up (7),(8),(9)and subsequently integrating with respect to the x-variable,we can obtain that the total energy

$$E(t) = \int_R u^2(t,x) + u_x^2(t,x) + \rho^2(t,x)\,dx \tag{10}$$

is constant in time. Notice that

$$|p * u| \le \frac{1}{2}\left(\|p\|_{L^2}^2 + \|u\|_{L^2}^2\right) \le \frac{1}{2}\left(\frac{1}{4} + \|u\|_{H^1}^2 + \|\rho\|_{L^2}^2\right) \le \frac{1}{2} + \frac{1}{2}E(0). \tag{11}$$

From the above bound on the total energy and (11) we have

$$\|P(t)\|_{L^\infty}, \|P_x(t)\|_{L^\infty} \le \left\|\tfrac{1}{2}e^{-|x|}\right\|_{L^\infty} \cdot \left\|u^2 + \frac{u_x^2}{2} + \frac{\rho^2}{2}\right\|_{L^1} \le \frac{1}{2}E(0). \tag{12}$$

In the following we will introduce an equivalent semilinear system. Let $\bar{z} = (\bar{u}\ ,\ \bar{\rho}) \in H^s \times H^{s-1},\ s > \frac{3}{2}$ be the initial data. Consider now the following initial problem

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t} q\left(t,\xi\right) = u\left(t, q\left(t,\xi\right)\right), & t \in [0\ ,\ T]\,, \\ q\left(0,\xi\right) = \bar{q}\left(\xi\right), & x \in R. \end{array} \right. \tag{13}$$

where u denotes the first component of the solution z to Eq. (14) remains Lipschitz continuous for $t \in [0, T]$, and let the non-decreasing map $\xi \mapsto \bar{q}\left(\xi\right)$ be defined by setting

$$\int_0^{\bar{q}(\xi)} \left(1 + \bar{u}_x^2\right) dx = \xi \tag{14}$$

then with the independent variables $(t\ ,\ \xi)$,we derive an equivalent system of equations. Moreover, we write $u\left(t,\xi\right) = u\left(t, q\left(t,\xi\right)\right)$, $\rho\left(t,\xi\right) = \rho\left(t, q\left(t,\xi\right)\right)$, $P\left(t,\xi\right) = P\left(t, q\left(t,\xi\right)\right)$, $P_x\left(t,\xi\right) = P_x\left(t, q\left(t,\xi\right)\right)$. We will further use the following variables:$v = v\left(t,\xi\right)$ and $w = w\left(t,\xi\right)$ defined as

$$v = 2\arctan u_x, w = \left(1 + u_x^2\right) \cdot \frac{\partial q}{\partial \xi}, \tag{15}$$

with $u_x\left(t,\xi\right) = u_x\left(t, q\left(t,\xi\right)\right)$. Remarkably v is defined up to multiples of2π (All subsequent equations involving are invariant under addition of multiples of 2π). And we know that $w\left(0,\xi\right) \equiv 1$. Applying classical results in the new variables, we can obtain some results as following:

$$P\left(t,\xi\right) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-\left|q(t,\xi) - q\left(t,\xi'\right)\right|} \left(u^2\left(t,x\right) + \frac{1}{2}u_x^2\left(t,x\right) + \frac{1}{2}\rho^2\left(t,x\right) - Au\left(t,x\right)\right) dx,$$

$$P_x\left(t,\xi\right) = \frac{1}{2}\left(\int_{q(t,\xi)}^{+\infty} - \int_{-\infty}^{q(t,\xi)}\right) e^{-\left|q(t,\xi) - q\left(t,\xi'\right)\right|} \left(u^2\left(t,x\right) + \frac{1}{2}u_x^2\left(t,x\right) + \frac{1}{2}\rho^2\left(t,x\right) - Au\left(t,x\right)\right) dx.$$

In the above formulae, we use the change of variables $y = q\left(t,\xi\right)\,,\ x = q\left(t,\xi'\right)$, and write the convolution as an integral over the variable ξ'. We also obtain an expression for P and P_x in terms of the new variableξ, namely

$$P\left(\xi\right) = \frac{1}{2}\int_{-\infty}^{+\infty} \exp\left\{-\left|\int_{\xi}^{\xi'} \cos^2\frac{v(s)}{2} \cdot w\left(s\right) ds\right|\right\} \cdot \left(\left(u^2 + \frac{1}{2}\rho^2 - Au\right)\cos^2\frac{v}{2} + \frac{1}{2}\sin^2\frac{v}{2}\right)\left(\xi'\right) w\left(\xi'\right) d\xi', \tag{16}$$

$$P_x\left(\xi\right) = \frac{1}{2}\left(\int_{\xi}^{+\infty} - \int_{-\infty}^{\xi}\right) \exp\left\{-\left|\int_{\xi}^{\xi'} \cos^2\frac{v(s)}{2} \cdot w\left(s\right) ds\right|\right\} \cdot \left(\left(u^2 + \frac{1}{2}\rho^2 - Au\right)\cos^2\frac{v}{2} + \frac{1}{2}\sin^2\frac{v}{2}\right)\left(\xi'\right) w\left(\xi'\right) d\xi'. \tag{17}$$

By (4) and (13), the evolution equation for u in the new variables (t, ξ) takes the form

$$\frac{\partial}{\partial t} u\left(t,\xi\right) = u_t + uu_x = -P_x\left(t,\xi\right) \tag{18}$$

with P_x given at (17). Using (13), (15) and (6), we obtain

$$\frac{\partial}{\partial t} v\left(t,\xi\right) = \frac{2}{1+u_x^2}\left(u_{xt} + uu_{xx}\right) = \frac{2}{1+u_x^2}\left(-\frac{u_x^2}{2} + u^2 - P + \frac{\rho^2}{2} - Au\right) = 2\left(u^2 - P + \frac{\rho^2}{2} - Au\right)\cos^2\frac{v}{2} - \sin^2\frac{v}{2}, \tag{19}$$

and from (13), (15) and (8), we have

$$\frac{\partial}{\partial t} w\left(t,\xi\right) = \left(2u^2 + \rho^2 + 1 - 2P - 2Au\right)\frac{u_x}{1+u_x^2} \cdot w = \left(u^2 + \frac{\rho^2}{2} + \frac{1}{2} - P - Au\right)\sin v \cdot w. \tag{20}$$

In the above (19) and (20), the function$P$ is computed by (16).

3 Global solutions of the semilinear system

Suppose initial data $\bar{z} = (\bar{u}\ ,\ \bar{\rho}) \in H^s \times H^{s-1},\ s > \frac{3}{2}$ be given. We consider the corresponding Cauchy problem with the variables (u, ρ, v, w) in the form

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = -P_x, \\ \frac{\partial \rho}{\partial t} = -\tan\frac{v}{2} \cdot \rho, \\ \frac{\partial v}{\partial t} = \left(u^2 - P + \frac{\rho^2}{2} - Au\right)\left(1 + \cos v\right) - \sin^2\frac{v}{2} \cdot w, \\ \frac{\partial w}{\partial t} = \left(u^2 + \frac{\rho^2}{2} + \frac{1}{2} - P - Au\right)\sin v \cdot w. \end{array} \right. \tag{21}$$

With initial data

$$\begin{cases} u\left(0,\xi\right)=\bar{u}\left(\bar{q}\left(\xi\right)\right), \\ \rho\left(0,\xi\right)=\bar{\rho}\left(\bar{q}\left(\xi\right)\right), \\ v\left(0,\xi\right)=2\arctan\bar{u}_x\left(\bar{q}\left(\xi\right)\right), \\ w\left(0,\xi\right)=1. \end{cases} \tag{22}$$

Here P , P_x are given by (16)-(17) as functions of u,ρ,v,w and ξ.We regard (21) as an ordinary differential equation in the Banach space $X=H^1\left(R\right)\times L^2\left(R\right)\times\left[L^2\left(R\right)\cap L^\infty\left(R\right)\right]\times L^\infty\left(R\right)$ with norm $\left\|(u,\rho,v,w)\right\|_X=\left\|u\right\|_{H^1}+\left\|\rho\right\|_{L^2}+\left\|v\right\|_{L^2}+\left\|v\right\|_{L^\infty}+\left\|w\right\|_{L^\infty}$.

Lemma 1 *Suppose that* $\bar{z}=(\bar{u}\ ,\ \bar{\rho})\in H^s\times H^{s-1},\ s>\frac{3}{2}$. *Then there exist* $T=T\left(\left\|\bar{z}\right\|_{H^s\times H^{s-1}}\right)$ *and a unique solution*$z=(u,\rho)\in C\left([0,T);H^s\times H^{s-1}\right)\cap C^1\left([0,T);H^{s-1}\times H^{s-2}\right)$ *of (4) with* $z\left(0\right)=\bar{z}$ *.Moreover, the solution* u *depends continuously on the initial value*$\bar{z}$ *and the maximal time of existence* $T>0$ *is independent of s .*

Lemma 2 *let* $\bar{z}=(\bar{u}\ ,\ \bar{\rho})\in H^s\times H^{s-1},\ s>\frac{3}{2}$, *then the corresponding solution z to (4) satisfies (i)* $\left\|u\left(t\ ,\ \cdot\right)\right\|_{L^\infty}^2\leq\frac{1}{2}\left(\left\|\bar{u}\right\|_{H^1}^2+\left\|\bar{\rho}-1\right\|_{L^2}^2\right),\quad\forall t\in[0,T)$, *(ii)* $\left\|\rho\left(t\ ,\ \cdot\right)\right\|_{L^\infty}\leq\left\|\bar{\rho}\right\|_{L^\infty}e^{C\int_0^t\left\|\partial_x u\right\|_{L^\infty}d\tau},\quad\forall t\in[0,T)$.

Lemma 3 *Given* $\bar{z}=(\bar{u}\ ,\ \bar{\rho})\in H^s\times H^{s-1},s>\frac{3}{2}$, *there exists a unique solution defined on some small time interval*$[0\ ,\ T]$ *with* $T>0$*for the Cauchy problem (21)-(22).*

Theorem 4 *Let* $\bar{z}=(\bar{u}\ ,\ \bar{\rho})\in H^s\times H^{s-1}$ *be as in lemma 3 with* $s>\frac{3}{2}$, *then there exists a unique global solution defined for all* $t\geq 0$ *for the Cauchy problem (21)-(22).*

Proof. To ensure that the local solution of (21) constructed above can be extended to a global solution, it is necessary to show that the quantity$\left\|u\right\|_{H^1}+\left\|\rho\left(t\right)\right\|_{L^2}+\left\|v\left(t\right)\right\|_{L^2}+\left\|v\left(t\right)\right\|_{L^\infty}+\left\|w\left(t\right)\right\|_{L^\infty}+\left\|\frac{1}{w(t)}\right\|$ remains uniformly bounded on any bounded time interval. Then , we re-derive the energy conservation property in terms of the new variables (u,ρ,v,w) and ξ .

Notice that $u_\xi=\frac{w}{2}\sin v$, we claim that

$$\begin{aligned} &u_{\xi t}=\left(\tfrac{w}{2}\sin v\right)_t=\tfrac{w_t}{2}\sin v+\tfrac{w}{2}v_t\cos v \\ &=\tfrac{w}{2}\left[\left(u^2+\tfrac{1}{2}-P+\tfrac{\rho^2}{2}\right)\sin^2 v+\left(u^2-P+\tfrac{\rho^2}{2}\right)\cos v+\left(u^2-P+\tfrac{\rho^2}{2}\right)^2\cos v-\cos v\cdot\sin^2\tfrac{v}{2}\right] \\ &=w\left[\left(u^2-P+\tfrac{\rho^2}{2}\right)\cdot\cos^2\tfrac{v}{2}+\tfrac{1}{2}\sin^2\tfrac{v}{2}\right]=w\left(u^2\cos^2\tfrac{v}{2}-P\cos^2\tfrac{v}{2}+\tfrac{\rho^2}{2}\cos^2\tfrac{v}{2}+\tfrac{1}{2}\sin^2\tfrac{v}{2}\right). \end{aligned} \tag{23}$$

Since the right-hand sides of (23)is Lipschitz continuous, we infer that u_ξ remains valid for all times $t\geq 0$, as long as the solution of (21)is defined.

From (21) we can obtain

$$\begin{aligned} &\tfrac{d}{dt}\int_R\left[u^2\cos^2\tfrac{v}{2}+\sin^2\tfrac{v}{2}+\left(\rho-1\right)^2\cos^2\tfrac{v}{2}\right]wd\xi \\ &=\int_R w\left\{2\sin\tfrac{v}{2}\cos\tfrac{v}{2}\left[u^2\cos^2\tfrac{v}{2}+\sin^2\tfrac{v}{2}+\left(\rho-1\right)^2\cos^2\tfrac{v}{2}\right]\left(u^2+\tfrac{1}{2}-P+\tfrac{\rho^2}{2}\right)-2uP_x\cos^2\tfrac{v}{2}\right. \\ &\left.-2\left(\rho-1\right)\tan\tfrac{v}{2}\cdot\rho\cdot\cos^2\tfrac{v}{2}+\sin\tfrac{v}{2}\cdot\cos\tfrac{v}{2}\cdot\left[1-u^2-\left(\rho-1\right)^2\right]\left[2\left(u^2-P+\tfrac{\rho^2}{2}\right)\cos^2\tfrac{v}{2}-\sin\tfrac{v}{2}\right]\right\}d\xi \\ &=\int_R w\left\{-2P\sin\tfrac{v}{2}\cdot\cos\tfrac{v}{2}-2uP_x\cos^2\tfrac{v}{2}+3u^2\sin\tfrac{v}{2}\cdot\cos\tfrac{v}{2}\right\}d\xi. \end{aligned} \tag{24}$$

On the other hand, in the sense of $P_\xi=wP_x\cos^2\frac{v}{2}$, $\left(uP\right)_\xi=u_\xi P+uP_\xi=w\left(P\sin\frac{v}{2}\cdot\cos\frac{v}{2}+uP_x\cos^2\frac{v}{2}\right)$, and $\left(u^3\right)_\xi=3u^2u_\xi=3wu^2\sin\frac{v}{2}\cdot\cos\frac{v}{2}$, the following identity can be observed

$$\tfrac{d}{dt}\int_R\left[u^2\cos^2\tfrac{v}{2}+\sin^2\tfrac{v}{2}+\left(\rho-1\right)^2\cos^2\tfrac{v}{2}\right]wd\xi=\int_R\partial_\xi\left(u^3-2uP\right)d\xi=0. \tag{25}$$

In terms of the new variables the total energy (10) can be rewritten as follows. According to (25), this energy remains constant in time,

$$E\left(t\right)=\int_R\left(u^2\left(t,\xi\right)\cos^2\tfrac{v(t,\xi)}{2}+\sin^2\tfrac{v(t,\xi)}{2}+\left(\rho-1\right)^2\cos^2\tfrac{v(t,\xi)}{2}\right)w\left(t,\xi\right)d\xi=E\left(0\right)=E_0. \tag{26}$$

From (26) it follows that

$$\left\|P\left(t\right)\right\|_{L^\infty},\left\|P_x\left(t\right)\right\|_{L^\infty}\leq\left\|\tfrac{1}{2}e^{-|x|}\right\|_{L^\infty}\cdot\left\|u^2\left(t\right)+\tfrac{u_x^2(t)}{2}+\tfrac{\rho^2(t)}{2}\right\|_{L^1}\leq\tfrac{1}{2}E_0. \tag{27}$$

The estimate (12) has been recovered, working in the new variables.

As long as the solution is defined, from the first equation in (21),we deduce that, $\left|\frac{d}{dt}\left(\int_R u^2(t,\xi)\right)\right| \le 2\left\|u(t)\right\|_{L^\infty}\left\|P_x(t)\right\|_{L^1}$, and $\left|\frac{d}{dt}\left(\int_R u_\xi^2(t,\xi)\right)\right| \le 2\left\|u_\xi(t)\right\|_{L^\infty}\left\|\partial_\xi P_x(t)\right\|_{L^1}$.

Moreover, the second equation in (21)implies $\left\|v(t)\right\|_{L^\infty} \le e^{Bt}$, for a suitable constant$B = B(E_0) > 0$. From the third equation in (21), (27) yields

$$|w_t| \le \left(E_0 + \tfrac{1}{2} + \tfrac{E_0}{2}\right)w, \quad \exp\left\{-\tfrac{1+3E_0}{2}t\right\} \le w(t) \le \exp\left\{\tfrac{1+3E_0}{2}t\right\}. \tag{28}$$

We have known that the functions u and u_ξ are uniformly bounded on bounded intervals of time in view of (28). If we callλ the right-hand side of (28), so that$\lambda^- \le w(t) \le \lambda^+$. We know that

$$\left\|\partial_\xi P(t)\right\|_{L^1} \leqslant \lambda E_0 + \tfrac{1}{2}\int_R \exp\left\{-\left|\int_\xi^{\xi'} \lambda^- \cos^2\tfrac{v(t,s)}{2}ds\right|\right\}$$
$$\cdot\left[u^2(t,\xi')\cos^2\tfrac{v(t,\xi)}{2} + \tfrac{1}{2}\sin^2\tfrac{v(t,\xi)}{2} + \tfrac{\rho^2(t,\xi')}{2}\cos^2\tfrac{v(t,\xi)}{2}\right]\lambda d\xi' \leqslant \lambda E_0 + \left\|\Lambda\right\|_{L^1}\cdot\lambda E_0.$$

Where $\Lambda(\zeta) = \min\left\{1, \exp\left(18E_0\lambda^{-1} - \frac{|\zeta|}{2}\lambda^{-1}\right)\right\}$, $\left\|\Lambda\right\|_{L^1} = \left(\int_{|\zeta|\le 36E_0} + \int_{|\zeta|\ge 36E_0}\right)\Lambda(\zeta)\,d\zeta = 72E_0 + \frac{4}{\lambda}$. The estimate for $\left\|P_x\right\|_{L^1}$ is entirely similar. This establishes the boundedness of the norm $\left\|u(t)\right\|_{H^1}$ for t in bounded intervals. The third equation in (21) also implies $\frac{d}{dt}\left\|v\right\|_{L^2} \le 2\left\|u\right\|_{L^\infty}\left\|u\right\|_{L^2} + 2\left\|P\right\|_{L^2} + \left\|\rho\right\|_{L^\infty}\left\|\rho\right\|_{L^2} + \frac{1}{4}\left\|v\right\|_{L^\infty}\left\|v\right\|_{L^2}$. Then $\left\|v\right\|_{L^2}$ remains bounded on bounded intervals of time. This completes the proof that the solution of (21) can be extended globally in time.

4 Global solutions to the two-component Camassa-Holm equation

In this section we will show that the global solution of the system (21) yields a global conservative solution to the two-component Camassa-Holm equation (4), in the original variables $(t\ ,\ x)$.

We begin with a global solution (u, ρ, v, w) to (21). Define

$$q(t,\xi) = \bar{q}(\xi) + \int_0^t u(\tau,\xi)\,d\tau. \tag{29}$$

For each fixed ξ, the function $t \mapsto q(t,\xi)$ provides a solution to the Cauchy problem

$$\tfrac{\partial}{\partial t}q(t,\xi) = u(t,\xi), q(0,\xi) = \bar{q}(\xi). \tag{30}$$

We claim that a solution of (4) can be obtained by setting

$$u(t,x) = u(t,\xi), if q(t,\xi) = x \tag{31}$$

Lemma 5 *let $\bar{z} = (\bar{u}, \bar{\rho}) \in H^s \times H^{s-1}, s > \frac{3}{2}$, and (u, ρ, v, w) provide a global solution to the Cauchy problem (13)-(14). Then we have*

$$\tfrac{\partial}{\partial t}q_\xi(t,\xi) = u_\xi(t,\xi) \tag{32}$$

for all $t \ge 0$ and a.e. $\xi \in R$.

Proof. Notice that $q_\xi = w\cos^2\frac{v}{2}$ for all $t \ge 0$ and a.e. $\xi \in R$, then we have

$$\begin{aligned}&\tfrac{\partial}{\partial t}q_\xi(t,\xi) = \tfrac{\partial}{\partial t}\left(w\cos^2\tfrac{v}{2}\right)(t,\xi) = -w\cdot\cos\tfrac{v}{2}\cdot\sin\tfrac{v}{2}\cdot v_t + w_t\cdot\cos^2\tfrac{v}{2}\\&= w\cos\tfrac{v}{2}\cdot\sin^3\tfrac{v}{2} - 2w\left(u^2 - P + \tfrac{\rho^2}{2}\right)\cos^3\tfrac{v}{2}\cdot\sin\tfrac{v}{2} + 2w\left(u^2 - P + \tfrac{\rho^2-1}{2}\right)\sin\tfrac{v}{2}\cdot\cos^3\tfrac{v}{2}\\&= w\cos\tfrac{v}{2}\cdot\sin^3\tfrac{v}{2} + w\cos^3\tfrac{v}{2}\cdot\sin\tfrac{v}{2} = w\sin\tfrac{v}{2}\cdot\cos\tfrac{v}{2} = w\cdot\tfrac{\sin v}{2} = u_\xi(t,\xi).\end{aligned} \tag{33}$$

This completes the proof of the lemma.

Theorem 6 *Suppose (u, ρ, v, w) provide a global solution to the Cauchy problem (13)-(14). Then the function $u = u(t,x)$ defined by (29)-(31) provides a global solution to the initial value problem (4)-(5) for the two-component Camassa-Holm equation. Moreover, the energy is almost always conserved, namely*

$$\left\|u\right\|_{H^1}^2 + \left\|\rho - 1\right\|_{L^2}^2 = \left\|\bar{u}\right\|_{H^1}^2 + \left\|\bar{\rho} - 1\right\|_{L^2}^2 \tag{34}$$

for a.e. $t \ge 0$.

Proof. Using the uniform bound $|u(t,\xi)| \le E_0^{1/2}$, from (29) it is clear that$\bar{q}(\xi) - E_0^{1/2} t \le q(t,\xi) \le \bar{q}(\xi) + E_0^{1/2} t, t \ge 0$. In view of the definition of ξ at (14), we can obtain $\lim\limits_{\xi \to \pm\infty} \bar{q}(t,\xi) = \pm\infty$.Then the image of the continuous map $(t,\xi) \to (t, q(t,\xi))$ is the entire plane R^2. It is clear that the map $\xi \to q(t,\xi)$ is non-decreasing. This proves that the map $(t,x) \to u(t,x)$ at (31) is well defined, for all$(t,x) \to u(t,x)$ and $x \in R$.

Next, for every fixed t, from (26) we can obtain

$$\begin{aligned} &\int_R \left(u^2(t,x) + u_x^2(t,x) + (\rho-1)^2\right) dx \\ &= \int_{\{\cos v > -1\}} \left(u^2(t,\xi)\cos^2\frac{v(t,\xi)}{2} + \sin^2\frac{v(t,\xi)}{2} + (\rho-1)^2(t,\xi)\cos^2\frac{v(t,\xi)}{2}\right) w(t,\xi)\, d\xi \le E_0. \end{aligned} \tag{35}$$

By a Sobolev inequality [8], this implies the uniform Holder continuity with exponent $\frac{1}{2}$ of u as a function of x. From the first equation in (21) and the bound $\|P_x\|_{L^\infty} \le \frac{E_0}{2}$, we obtain that the map $t \mapsto u(t, q(t))$ is uniformly Lipschitz continuous along every characteristic curve$t \mapsto q(t)$. Therefore,$u = u(t,x)$ is globally H?lder continuous on the entire(t,x) half plane. This completes the proof that$z = (u,\rho)$ is a global solution of the two-component Camassa-Holm equation in the sense of Definition 1.

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