Well-posedness of Higher-order Dullin-Gottwald-Holm Equations

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Abstract: In this paper, the well-posedness of higher-order Dullin-Gottwald-Holm equations (in the following we call it DGH equations) is investigated. We establish the existence of global weak solutions. Finally, we extend our results to a generalized case.

Keywords: well-posedness; higher-order Dullin-Gottwald-Holm equations; weak solutions

1 Introduction

Dullin, Gottwald and Holm in [1] discussed the following nonlinear equation

$$m_t + 2\omega u_x + um_x + 2mu_x = -\gamma u_{xxx}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R},$$

(1)

where $m = u - \alpha^2 u_{xx}$ is a momentum variable, $u(x,t)$ is the fluid velocity. The constants $\alpha^2$ and $\frac{\omega}{c_0}$ are squares of length scales, and $c_0 = \sqrt{gh}$ ($c_0 = 2\omega$) is the linear wave speed for undisturbed water at rest at spatial infinity. Equation (1) was derived by using asymptotic expansions directly in the Hamiltonian for Euler’s equations in the shallow water regime. It is the so-called DGH equation that we are interested in. Eq. (1) is a new 1+1 quadratically equation for a unidirectional shallow water wave. It is completely integrable and its traveling wave solution contains both Korteweg-de Vries (KdV) solitons and Camassa-Holm (CH) peakons as limiting case. For $\alpha \rightarrow 0$, Eq. (1) becomes KdV equation

$$u_t + 2\omega u_x + 3uu_x = -\gamma u_{xxx}.$$ When $\gamma = 0$, Eq. (1) turns out to be the CH equation

$$u_t - \alpha^2 u_{xxt} + 2\omega u_x + 3uu_x = \alpha^2(2u_x u_{xx} + uu_{xxx}).$$

(2)

KdV equation describes the unidirectional propagation of waves at free surface of shallow water under the influence of the gravity. CH equation appears first as a bi-Hamiltonian equation with infinite number of conserved functionals. The solitary waves of the KdV equation are smooth solitons. The solitary waves of CH are smooth for $\omega > 0$ and peaked for $\omega = 0$.

In [2], the local well-posedness for (1) with initial datum $u_0 \in H^s(\mathbb{R}), \ s > \frac{3}{2}$ was established by applying Kato’s theory. Tian, Gui, Liu in [2,6-7] proved that Eq. (1) had a global solution and its solitary waves were stable. They also obtained the locally strong limit of the solution as the dispersive parameter $\gamma$ tended to zero and demonstrated that its solutions converged to the solution of the corresponding KdV equation as $\alpha^2$ tended to zero. Using a direct method, they got the new exact peaked solitary wave solutions of the DGH equation. Eq. (1) is shown to be bi-Hamiltonian and has a Lax pair formulation. The Cauchy problem of Eq. (1) (for $\omega = 0$) has been well studied. Particularly, in [3], G.M. Coclite, H. Holden and K.H. Karlsen studied the well-posedness of higher-order Camassa-Holm equations. However, there have been no researches on the well-posedness of higher-order Dullin-Gottwald-Holm equations. So in this paper we employ the singular perturbation approach to study the problem of the well-posedness of higher-order Dullin-Gottwald-Holm equations.

The structure of the paper is organized as follows. In Section 2, we construct a family of higher-order DGH equations. We give an important result about $H^k$-norm and $L^\infty$-norm of $u_\varepsilon$ and we get a series of norm estimates about $p_\varepsilon$ in Section 3, so we obtain the existence of the weak solutions and get the main theorems. In Section 4, we deal with the general case of $k > 2$.

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2 The higher-order DGH equations

We know that the Cauchy problem of the DGH equation is as follows

\[
\begin{cases}
    u_t - \alpha^2 u_{xxtt} + 2\omega u_x + 3uu_x + \gamma u_{xxt} = \alpha^2(2u_xu_{xx} + uu_{xxx}), & t > 0, \ x \in R, \\
    u(0, x) = u_0(x), \ x \in R.
\end{cases}
\]  

From the method of \[3,5\], we can construct a family of higher-order DGH equations. For simplicity, we only discuss the case of \(\alpha^2 = 1\), we consider the equation \(\partial_t u = B_k(u, u)\), \(k \in N_0 = N \cup \{0\}\), where \(u = u(t, x) : [0, \infty) \times R \to R\) is an unknown function,

\[
A_k(u) = \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \quad B_k(u, u) = A_k^{-1}(C_k(u)) - u\partial_x u,
\]

\[
C_k(u) = -(u + 2\omega)A_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x uA_k(u) - (2\omega + \gamma)\partial_x^3 u.
\]

**Remark 1** The operator \(A_k^{-1}\) has a convolution structure, \(A_k^{-1}(f)(x) = \int_R G_k(x - y)f(y)dy, x \in R\), where \(G_k\) has Fourier transform \(G_k\) given by \(G_k(\xi) = \frac{1}{1 + \omega^2 + \xi^2}\), \(\xi \in R\). In the special case we have \(G_1(1) = \frac{1}{2}e^{-|x|}\), and we also have

\[
G_k \geq 0, \quad \|G_k\|_{W^{k-1,\infty}(R)} \leq C_0, \quad (C_0 \text{ is a positive constant}).
\]

From the operator of \(C_k(u)\), we can denote \(C_k(u) = -\partial_x F_k(u), \text{i.e.} F_k(u) = -\int_{-\infty}^{x} C_k(u_\xi)d\xi\).

Through the concrete calculation of the cases \(k = 0, 1, 2, 3\), we can get the general case of the higher-order DGH equations

\[
\begin{cases}
    \partial_t u + u\partial_x u + \partial_x p = 0, \\
    A_k(p) = F_k(u),
\end{cases}
\]

which is equivalent to the equations

\[
\begin{cases}
    \partial_t u = B_k(u, u), \\
    u(0, x) = u_0(x).
\end{cases}
\]

We will assume

\[
u_0 \in H^k(R), \quad \partial_x^k u_0 \in L^p(R), \quad 2 < p < \infty.
\]

**Definition 1** We call a function \(u : [0, \infty) \times R \to R\) a weak solution of (6) if i) satisfies (5) in the sense of distributions; ii) \(u \in C([0, \infty); C^{k-1}(R)) \cap L^\infty([0, \infty); H^k(R)); iii) u(x, 0) = u_0(x)\) for every \(x \in R\); iv) \(\|u(t, \cdot)\|_{H^{k+1}(R)} \leq \|u_0\|_{H^k(R)}\) for each \(t > 0\).

We apply the following singular perturbation approach to get the approximant solution of the equations (6). Let \(\varepsilon > 0\) and consider the system

\[
\begin{cases}
    \partial_t u + u\partial_x u + \partial_x p = \varepsilon \partial_x^2 u, & t > 0, \ x \in R, \\
    A_k(p) = F_k(u), & t > 0, \ x \in R, \\
    u(x, 0) = u_{0, \varepsilon}(x), \ x \in R.
\end{cases}
\]

where

\[
u_{0, \varepsilon} \in H^{k+1}(R), \quad \|u_{0, \varepsilon}\|_{H^{k+1}(R)} \leq \|u_0\|_{H^k(R)},
\]

\(u_{0, \varepsilon} \to u_0\) in \(H^k(R)\). We call the solution \(u_{\varepsilon} = u_{\varepsilon}(t, x)\) of (8) a viscous approximant to the solution \(u = u(t, x)\) of (6). That is \(u_{\varepsilon} \to u \text{ as} \varepsilon \to 0\). Employing the method above, Eqs. (8) read in the special cases \(k = 0, 1, 2, 3\) as follows:

i) For \(k = 0\), we find the following

\[
\begin{align*}
    \partial_t u + u\partial_x u + \partial_x p &= \varepsilon \partial_x^2 u, \\
p_c &= u_x^2 + 2\omega u_x + (2\omega + \gamma)\partial_x^2 u_x.
\end{align*}
\]

ii) For \(k = 1\), we find \(\partial_t u + u\partial_x u + \partial_x p = \varepsilon \partial_x^2 u, \partial_x p - \partial_x^2 p = u_x^2 + \frac{1}{2}(\partial_x u_x)^2 + 2\omega u_x + \gamma \partial_x^2 u_x.
\]

iii) For \(k = 2\), we have \(\partial_t u + u\partial_x u + \partial_x p = \varepsilon \partial_x^2 u, \partial_x p - \partial_x^2 p = u_x^2 + \frac{1}{2}(\partial_x u_x)^2 - \frac{7}{2}(\partial_x^2 u_x)^2 - 3\partial_x u_x \partial_x^3 u_x + 2\omega u_x + 2\omega \partial_x^2 u_x + \gamma \partial_x^2 u_x.
\]

iv) For \(k = 3\), we get \(\partial_t u + u\partial_x u + \partial_x p = \varepsilon \partial_x^2 u, \partial_x p - \partial_x^2 p + \partial_x^2 p = u_x^2 + \frac{1}{2}(\partial_x u_x)^2 - \frac{7}{2}(\partial_x^2 u_x)^2 - 3\partial_x u_x \partial_x^3 u_x + \frac{19}{2}(\partial_x^4 u_x)^2 + 16\partial_x^2 u_x^2 \partial_x^2 u_x + 5\partial_x u_x \partial_x^3 u_x + 2\omega u_x - 2\omega \partial_x^2 u_x + \gamma \partial_x^2 u_x.
\]
3 The main lemmas and our result

Lemma 2 (Global existence) Under the assumptions of (4) and (9), let \( \varepsilon > 0 \), then there exists a unique global smooth solution \( u_\varepsilon = u_\varepsilon(t, x) \) of the Cauchy problem (8) belonging to \( C([0, \infty); H^{k+1}(R)) \).

Lemma 3 (Energy estimate) Under the assumptions of (4) and (9), the identity

\[
\|u_\varepsilon(t, \cdot)\|_{H^k(R)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\tau, \cdot)\|_{H^k(R)}^2 d\tau = \|u_0, \varepsilon\|_{H^k(R)}^2 \leq \|u_0\|_{H^k(R)}^2 \tag{11}
\]

holds for each \( t \geq 0 \) and \( \varepsilon > 0 \). In addition,

\[
\|u_\varepsilon\|_{L^\infty([0, \infty) \times R)}, \ldots, \|\partial_x^{k-1} u_\varepsilon\|_{L^\infty([0, \infty) \times R)} \leq \frac{1}{\sqrt{2}} \|u_0\|_{H^k(R)} \tag{12}
\]

for each \( \varepsilon > 0 \).

Proof. The proofs of these two lemmas are respectively similar to the references of [3] and [4], so we omit them here.

From now on we assume \( k = 2 \). According to the identity \( A_k(p_\varepsilon) = F_k(u_\varepsilon) \), we have \( p_\varepsilon = A_k^{-1}(F_k(u_\varepsilon)) \). Then

\[
p_\varepsilon = A_2^{-1}(F_2(u_\varepsilon)) = A_2^{-1}(u_\varepsilon^2 + \frac{1}{2} u_\varepsilon^2 - \frac{7}{2} (\partial_x^2 u_\varepsilon)^2 - 3 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon + 2\omega u_\varepsilon + 2\omega \partial_x^2 u_\varepsilon + \gamma \partial_x^2 u_\varepsilon).
\]

Let \( p_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon} + p_{3,\varepsilon} \), where

\[
p_{1,\varepsilon}(t,x) = \int_R G_2(x-y)\left[u_\varepsilon^2(t,y) + \frac{1}{2} u_\varepsilon^2(t,y) - \frac{1}{2} (\partial_x^2 u_\varepsilon(t,y))^2\right] dy,
\]

\[
p_{2,\varepsilon}(t,x) = -3 \int_R G_2(x-y)\left[(\partial_x^2 u_\varepsilon)^2 + \partial_x u_\varepsilon \partial_x^2 u_\varepsilon\right] dy, p_{3,\varepsilon}(t,x) = \int_R G_2(x-y)\left[2\omega u_\varepsilon + 2\omega \partial_x^2 u_\varepsilon + \gamma \partial_x^2 u_\varepsilon\right] dy.
\]

Moreover, since \( G_2 \) is the Green’s function of the operator \( A_2 \), we have

\[
\partial_x^2 p_{2,\varepsilon}(t,x) = -3 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon - 3 \int_R \left(G_2^2(x-y) - G_2(x-y)\right) \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy.
\]

Let \( p_{4,\varepsilon}(t,x) = -3 \int_R \left(G_2^2(x-y) - G_2(x-y)\right) \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy, \) hence \( \partial_x^2 p_\varepsilon = \partial_x^2 p_{1,\varepsilon} - 3 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon + p_{4,\varepsilon} + \partial_x^2 p_{3,\varepsilon} \), for each \( \varepsilon > 0 \), \( t > 0 \), \( x \in R \).

Lemma 4 Assume \( k=2 \), (4) and (9) hold, we have the following inequalities

\[
\|p_\varepsilon(t, \cdot)\|_{W^{2,1}(R)} \leq 6C_0 \|u_0\|_{H^2(R)} + 3C_0(2\omega + \gamma) \|u_0\|_{H^2(R)} , \tag{13}
\]

\[
\|p_{1,\varepsilon}(t, \cdot)\|_{W^{4,1}(R)} \leq (6C_0 + 1) \|u_0\|_{H^2(R)} , \tag{14}
\]

\[
\|p_{2,\varepsilon}(t, \cdot)\|_{W^{2,1}(R)} \leq 3C_0 \|u_0\|_{H^2(R)} , \tag{15}
\]

\[
\|\partial_x^2 p_\varepsilon(t, \cdot)\|_{L^1(R)} \leq (7C_0 + 3) \|u_0\|_{H^2(R)} + C_0(4\omega + \gamma) \|u_0\|_{H^2(R)} , \tag{16}
\]

\[
\|p_{3,\varepsilon}(t, \cdot)\|_{W^{2,1}(R)} \leq 3C_0(4\omega + \gamma) \|u_0\|_{H^2(R)} , \tag{17}
\]

\[
\|p_{4,\varepsilon}(t, \cdot)\|_{W^{2,1}(R)} \leq 12C_0 \|u_0\|_{H^2(R)}^2 . \tag{18}
\]

Proof. The proof of (14) and (18) can be seen in [3], so we only need to prove the others. Using

\[
\partial_x^2 p_{2,\varepsilon}(t,x) = -3 \int_R \frac{d^2G_2(x-y)}{dx^2}(\partial_x^2 u_\varepsilon)^2 + \partial_x u_\varepsilon \partial_x^3 u_\varepsilon\] 

\[
= -3 \int_R \frac{d^2G_2(x-y)}{dx^2}(\partial_x u_\varepsilon) d(\partial_x u_\varepsilon \partial_x^2 u_\varepsilon) = -3 \int_R \frac{d^3G_2(x-y)}{dx^3}(\partial_x u_\varepsilon) \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy.
\]
Employing the Minkowski inequality, Hölder inequality, (4) and (9), we have

$$\|\partial_x^kp_z\|_{L^p_t(L^r)} \leq \|\partial_x^{k+1}G_2\|_{L^p_t(L^r)} \int_R |\partial_x u_x \partial_x^2 u_z| \, dy \leq C_0 \|\partial_x u_x\|_{L^2} \|\partial_x^2 u_z\|_{L^2} \leq C_0 \|u_0\|_{H^2(R)}^2.$$ 

for each $p \in \{1, \infty\}, i \in \{0, 1, 2\}$. Then we have (15). Now we prove (17). Since

$$\partial_t^j\partial_x^k u_z(t, x) = \int_R \frac{dG_2}{dx}(x - y)(2\omega u_x + 2\partial_x^2 u_z + \gamma \partial_x^3 u_z) \, dy,$$ 

employing the Minkowski inequality, Hölder inequality, we have

$$\|\partial_x^j\partial_x^k u_z\|_{L^p_t(L^r)} \leq \|\frac{dG_2}{dx}\|_{L^p_t(L^r)} \int_R |2\omega u_x + 2\partial_x^2 u_z + \gamma \partial_x^3 u_z| \, dy \leq C_0(2\omega \|u_x\|_{H^2(R)} + 2\omega \|\partial_x^2 u_z\|_{H^2(R)} + \gamma \|\partial_x^3 u_z\|_{H^2(R)}) \leq C_0(4\omega + \gamma) \|u_0\|_{H^2(R)}.$$ 

for each $i \in \{0, 1, 2\}$. This proves (17). From $\partial_x^2 p_z = \partial_x^2 p_{4, z} - 3\partial_x u_x \partial_x^2 u_z + \partial_x^4 p_{z}$, we have

$$\|\partial_x^2 p_z\|_{L^1_t(L^r)} \leq 3 \|u_0\|_{H^2(R)}^2 + C_0 \|u_0\|_{H^2(R)}^3 + C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)} + 6C_0 \|u_0\|_{H^2(R)}^2 = (7C_0 + 3) \|u_0\|_{H^2(R)}^2 + C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)}.$$ 

From (14), (15), (17), we can easily get (13).

**Lemma 5** Assume $k = 2$, (4) and (9) hold, then the following inequalities hold

$$\|\partial_t u_x\|_{L^2_t(L^r)} \leq \frac{1}{\sqrt{2}} \|u_0\|_{H^2(R)}^2 + 2C_0 \|u_0\|_{H^2(R)}^2 + C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)} + \varepsilon \|u_0\|_{H^2(R)},$$

$$\|\partial_t \partial_x u_x\|_{L^2_t(L^r)} \leq \sqrt{2T} \|u_0\|_{H^2(R)}^2 + (2C_0 \|u_0\|_{H^2(R)}^2 + C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)} + \varepsilon \|u_0\|_{H^2(R)} \sqrt{T} + \frac{\varepsilon}{\sqrt{2}} \|u_0\|_{H^2(R)} \sqrt{T},$$

where $T = [0, T] \times R, \ T > 0, \ t > 0, \ 0 < \varepsilon < 1$.

**Proof.** From the identity (10) $\partial_t u_x = -u_x \partial_x u_x - \partial_x p_z + \varepsilon \partial_x^2 u_z$ and Hölder inequality, we have

$$\|\partial_t u_x(t, \cdot)\|_{L^2_t(L^r)} \leq \frac{1}{\sqrt{2}} \|u_0\|_{H^2(R)}^2 + 2C_0 \|u_0\|_{H^2(R)}^2 + C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)} + \varepsilon \|u_0\|_{H^2(R)}.$$ 

Furthermore, differentiating (10) with respect to $x$, we get

$$\partial_t \partial_x u_x + (\partial_x u_x)^2 + u_x \partial_x^2 u_x + \partial_x^2 p_z = \varepsilon \partial_x^3 u_x,$$

hence

$$\|\partial_t \partial_x u_x(t, \cdot)\|_{L^2_t(L^r)} \leq \frac{1}{\sqrt{2}} \|u_0\|_{H^2(R)} \sqrt{T} + \|u_x\|_{L^\infty_t(L^r)} \|\partial_x^2 u_x\|_{L^2_t(L^r)} + 2C_0 \|u_0\|_{H^2(R)} \sqrt{T}$$

$$+ C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)} \sqrt{T} + \frac{\varepsilon}{\sqrt{2}} \|u_0\|_{H^2(R)} \sqrt{T}.$$ 

**Remark 6** Assume $k = 2$, (4) and (9) hold, Let $T > 0, \ v > 0$, we can get some positive constants depending only on $\|u_0\|_{H^2(R)}$, $T$ and independent of $v$, such that $\|\partial_t \partial_t^2 p_z\|_{L^1_t(L^r)} \|\partial_t \partial_t^2 p_z\|_{L^1_t(L^r)} \|\partial_t^3 p_z\|_{L^1_t(L^r)}$, $\|\partial_t^3 p_z\|_{L^1_t(L^1_t)}$, $\|\partial_t^2 u_z(t, \cdot)\|_{L^2_t(L^r)}$, $\|\partial_t^2 u_z(t, \cdot)\|_{L^2_t(L^r)}$, $\|\partial_t^2 u_z(t, \cdot)\|_{L^2_t(L^r)}$, $\|\partial_t^2 u_z(t, \cdot)\|_{L^2_t(L^r)}$, $\|\partial_t^3 u_z(t, \cdot)\|_{L^2_t(L^r)}$, and $\|\partial_t^3 u_z(t, \cdot)\|_{L^\infty_t(L^r)}$ are all uniformly bounded.

**Lemma 7** Assume $k = 2$, (4) and (9) hold, then the inequalities hold

$$\|\partial_t^2 p_z\|_{L^2_t(L^r)} \leq (2C_0 \|u_0\|_{H^2(R)} + C_0 (4\omega + \gamma) \|u_0\|_{H^2(R)}) \sqrt{T},$$

$$\|\partial_t^2 u_z(t, \cdot)\|_{L^2_t(L^r)} \leq 16 \|u_0\|_{H^2(R)}^2 T + (2C_0 \|u_0\|_{H^2(R)}^2 + (C_0 (4\omega + \gamma) + \varepsilon) \|u_0\|_{H^2(R)} \sqrt{T}.$$
we can get the first inequality, and
\[ L \text{ is uniformly bounded, where Lemma 11} \]

From (19) we can easily get (20).

**Lemma 8** Assume \( k = 2, (4) \) and (9) hold. There exists a constant \( K > 0 \) depending only on \( \| u_0 \|_{H^2(R)} \), \( \omega \) and \( \gamma \) but independent of \( \varepsilon \), such that \( \| \partial_t \partial_x^k p_c \|_{L^1(\mathbb{R}^+)} \leq K \varepsilon \) for each \( i \in \{0, 1, 2\} \), \( T \geq 0 \) and \( \varepsilon > 0 \).

**Proof.** We can rewrite \( p_c \) as \( p_c = p_{5, \varepsilon} + p_{6, \varepsilon} + p_{7, \varepsilon} \), where \( p_{5, \varepsilon} = \int_R G_2(x-y) \left[ \frac{1}{2} (\partial_x u_{\varepsilon})^2 + u_{\varepsilon}^2 \right] dy \),

\[ p_{6, \varepsilon}(t, x) = -\int_R G_2(x-y) \left[ \frac{7}{2} (\partial_x^2 u_{\varepsilon})^2 + 3 \partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon} \right] dy \]

and \( p_{7, \varepsilon}(t, x) = \int_R G_2(x-y) \left[ 2 \omega u_{\varepsilon} + 2 \omega \partial_x^2 u_{\varepsilon} + \gamma \partial_x^2 u_{\varepsilon} \right] dy \).

From the reference [3], we know
\[ \| \partial_t \partial_x^k p_c \|_{L^1(\mathbb{R}^+)} \leq c_1 (1 + T), \]
where \( c_1 > 0 \) is a constant depending only on \( \| u_0 \|_{H^2(R)} \).

For some constant \( c_2 > 0 \) (depending only on \( \| u_0 \|_{H^2(R)} \)). Then we only need to estimate \( \| \partial_t \partial_x^k p_7 \|_{L^1(\mathbb{R}^+)} \).

Employing Lemma 7, (21) and (22), we can get the result.

From the above estimates and the reference [3], we can derive two important results:

**Lemma 9** Let \( 2 < p < \infty \). Assume that \( u_0 \in H_{2,p} \). Then the family \( \{ u_c \}_{\varepsilon > 0} \) that solves (8) for \( k = 2 \) is compact in \( L^\infty_\text{loc}(0, \infty; H^2(R)) \). Thus there exist a positive sequence \( \{ \varepsilon_n \}_{n \in \mathbb{N}} \) decreasing to 0 and a function \( u \in L^\infty([0, \infty); H^2(R)) \cap H^1([0, T]; H^1(R)) \), for each \( T > 0 \), such that \( \varepsilon_{n+1} \to u \) in \( L^\infty([0, \infty); H^2(R)) \), for each \( T > 0 \); (ii) \( u \) is a weak solution of (6) for \( k = 2 \).

**Lemma 10** Assume \( k = 2 \). Let \( u_1, u_2 \) be two weak solutions of the system (6) in the sense of Definition 1. If there exists a map \( b \in L^1([0, T]) \), \( T > 0 \), such that
\[ \| \partial_x^2 u_1(t, \cdot) \|_{L^\infty(R)} + \| \partial_x^2 u_2(t, \cdot) \|_{L^\infty(R)} \leq b(t), \quad t \geq 0 \]
then \( L(t) \leq L(0) + c \int_0^t (1 + b(s)) L(s) ds \), for each \( t \geq 0 \) and some constant \( c > 0 \), where
\[ L(t) = \| w(t, \cdot) \|_{L^\infty(R)} + \| \partial_x w(t, \cdot) \|_{L^\infty(R)} + \| (\partial_x^2 - \partial_x) A_2^{-1} e(t, \cdot) \|_{L^\infty(R)} + \| A_2^{-2} e(t, \cdot) \|_{L^\infty(R)} \]
and the notation
\[ w = u_1 - u_2, \quad v = u_1 + u_2, \quad e = u_1^2 + (\partial_x^2 u_1)^2 + (\partial_x^2 u_2)^2, \quad e = \frac{1}{2} (e_1 - e_2). \]

From the above lemmas, we can get the existence and uniqueness of solutions to the Cauchy problem of (6).

## 4 The general case of \( k > 2 \)

Firstly, we prove the consistency of the weak formulation for a general \( k \).

**Lemma 11** (see [31]) Assume \( k \geq 2 \), \( f \in C^\infty([0, \infty) \times R) \) and \( \phi \in C^\infty([0, \infty) \times R) \). Then
\[ I_1 = \int_{[0, \infty) \times R} \int_t^\infty (A_k(f \partial_x f) - 2 \partial_x f A_k(f) - f A_k(\partial_x f))(t, y) \phi(t, x) dt dx < \infty. \]
\[ I_2 = \int_{[0, \infty) \times R} \int_t^\infty (2 \omega A_k(\partial_x f(t, y)) + (2 \omega + \gamma) \partial_x^2 f(t, y)) \phi(t, x) dt dx \]
is uniformly bounded, where \( j = 0, 1, \ldots, k \).
**Remark 12** Let \( f \in C^\infty([0,\infty) \times R) \) and \( k \geq 2 \), from (23) and (25), we can get 
\[
\int_{[0,\infty) \times R} F_k(f) \phi dt dx < \infty
\]
for any \( \phi \in C_c^\infty([0,\infty) \times R) \). Then we have \( f \in L^\infty([0,\infty); H^k(R)) \).

Next, we present that the ideas used in the above sections can also be applied in the general case of \( k > 2 \). From lemma 3, we get the boundedness of the family \( \{u_\varepsilon\} \), \( \varepsilon > 0 \) in \( L^\infty([0,\infty); H^k(R)) \). We can also prove that the family of \( \{\partial_x^k u_\varepsilon\} \), \( \varepsilon > 0 \) is compact in \( L^\infty([0,\infty); L^2(R)) \). We introduce the notation \( q_\varepsilon = \partial_x^k u_\varepsilon \), from calculation, we can get
\[
\partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + \beta \partial_x u_\varepsilon \partial_x^k u_\varepsilon - 2\omega (-1)^k \partial_x^k q_\varepsilon + M_\varepsilon = 0,
\]
where \( \beta = k + 1 + (-1)^k(2k - 1) \), \( M_\varepsilon = \Omega_\varepsilon + \overline{\rho_\varepsilon} \Omega_\varepsilon \) and \( \overline{\rho_\varepsilon} \) are all uniformly bounded in \( L^\infty([0,\infty); L^1(R) \cap L^\infty(R)) \). This general case can be applied to the particular case. Hence we complete the extension.

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**References**


