Uniqueness Results for a Class of Semilinear Elliptic Systems

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Abstract: We prove uniqueness of positive solution for the systems $-\Delta u = \lambda f(v) + m\psi_p(v)$, $-\Delta v = \mu g(u) + m\psi_q(u)$ in $\Omega$, $u = v = 0$ on $\partial \Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\psi_r(z) = |z|^{r-1}z$. $f, g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $f(x) \sim x^{p-1}$, $g(x) \sim x^{q-1}$ at $\infty$ for some positive numbers $p, q$ with $pq < 1$, and $\lambda, \mu$ are large positive parameters and $m \geq 0$.

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1 Introduction

Consider the boundary value problem
\begin{align*}
-\Delta u &= \lambda f(v) + m\psi_p(v); \quad x \in \Omega \\
-\Delta v &= \mu g(u) + m\psi_q(u); \quad x \in \Omega \\
u(x) &= v(x) = 0 \quad x \in \partial \Omega
\end{align*}
(1)
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ and $\lambda, \mu$ are large positive parameters, $m \geq 0$ and $f, g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$.

In the literature many results focus on the existence and uniqueness of positive solutions to boundary value problems, and we refer the reader to the papers [2-6, 11-14]. For example, In [2,6] the authors consider the existence of positive solutions for the p-Laplacian system with large $\lambda$:
\begin{align*}
-\Delta_p u &= \lambda f(v) \quad \text{in} \ \Omega \\
-\Delta_p v &= \lambda g(u) \quad \text{in} \ \Omega \\
u &= v = 0 \quad \text{on} \ \partial \Omega
\end{align*}
(2)
and
\begin{align*}
-\Delta u &= \lambda f(v) \quad \text{in} \ \Omega \\
-\Delta v &= \lambda g(u) \quad \text{in} \ \Omega \\
u &= v = 0 \quad \text{on} \ \partial \Omega
\end{align*}
(3)

In [3], Dalmasso proved existence and uniqueness of positive solutions to (3) when the composition $f(\chi g)$ is sublinear at $\infty$ and superlinear at 0 for each $c > 0$. The arguments in [5] rely on the fact the positive solutions to (3) in a ball are radially symmetric and decreasing (see [4,10]). D.D.Hai in [1] discussed about the uniqueness of positive solutions for semilinear elliptic system (3).

In this paper we shall extend the results in [1] and [5] to the case of a bounded domain in $\mathbb{R}^N$ and for a class of Laplacian systems. Results for the single equation case were obtain in [7,9]. Our approach is based on sub and super-solution, and maximum principle, and weak comparison principle.

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2 Existence and uniqueness results

We make the following assumptions

(H.1) $f, g : R^+ \rightarrow R^+$ are nondecreasing, continuous, $C^1$ on $(0, \infty), \limsup_{x \to 0^+} xf(x) < \infty$, $\limsup_{x \to 0^+} xg(x) < \infty$

(H.2) There exist positive $\beta, \delta, p, q$ with pq < 1 such that $\beta x^{p-1} \leq f(x) \leq \delta x^{q-1}$, $\beta x^{q-1} \leq g(x) \leq \delta x^{q-1}$, for all $x \geq 0$, and for $p_1 > p, q_1 > q$, $\frac{f(x)}{x^{p_1}}$ and $\frac{g(x)}{x^{q_1}}$ are nonincreasing for $x$ large.

Our main result is

Theorem 1 Let (H.1) – (H.2) hold. Then system (1) has a unique positive solution for $\min((\lambda \delta + m)(\mu \delta + m)^{p-1}, (\lambda \delta + m)^{q-1}(\mu \delta + m))$ large.

The next Lemma provides estimates for solutions to (1). When $\Omega$ is a ball, it was established in [5]. We shall denote the norm in $C^k(\overline{\Omega})$ by $|.|_k$.

Lemma 2 Let $(u, v)$ be a positive solution of (1). Then there exist positive $M$ and $M_i$, for $1 \leq i \leq 4$, such that

$$M_1((\lambda \delta + m)(\mu \delta + m)^{p-1})^{\frac{1}{p+\rho}} d(x, \partial \Omega) \leq u(x) \leq M_2((\lambda \delta + m)(\mu \delta + m)^{p-1})^{\frac{1}{p+\rho}} d(x, \partial \Omega)$$

$$M_3((\lambda \delta + m)^{q-1}(\mu \delta + m))^{\frac{1}{p+\rho}} d(x, \partial \Omega) \leq v(x) \leq M_4((\lambda \delta + m)^{q-1}(\mu \delta + m))^{\frac{1}{p+\rho}} d(x, \partial \Omega)$$

for $\min((\lambda \delta + m)(\mu \delta + m)^{p-1}, (\lambda \delta + m)^{q-1}(\mu \delta + m)) > M$. Here $d(x, \partial \Omega)$ denotes the distance for $x$ to $\partial \Omega$.

Proof. Let $(u, v)$ be a positive solution for (1). We first establish the upper estimate for $v$. In what follows, we shall denote by $C_i$ positive constant independent of $\lambda, \mu, u, v$. Using the equations for $u$ and $v$, we obtain

$$u(x) = \int_{\Omega} \lambda K(x, y) f(v(y)) + m \psi_p(v(y)) dy,$$

$$v(x) = \int_{\Omega} \mu K(x, y) g(u(y)) + m \psi_q(u(y)) dy$$

where $K(x, y)$ denotes the Green’s function of $-\Delta$ with Dirichlet boundary conditions. Thus, by (H.2),

$$-\Delta u = \lambda f(v) + m \psi_p(v) \leq (\lambda \delta + m) v^{p-1}$$

(4)

and with follow from weak comparison principle we have

$$|u|_0 \leq C(\lambda f(|v|_0) + m \psi_p(|v|_0) \leq C_1 |v|_0^{p-1}(\lambda \delta + m)$$

(5)

and

$$-\Delta v = \mu g(u) + m \psi_q(u) \leq (\mu \delta + m) u^{q-1}$$

(6)

and

$$|v|_0 \leq C(\mu g(|u|_0) + m \psi_q(|u|_0)) \leq C_1 |u|_0^{q-1}(\mu \delta + m)$$

(7)

From (5), (7), it follow that

$$|u|_0 \leq C_2((\mu \delta + m)^{p-1}(\lambda \delta + m))^\frac{1}{p+\rho}$$

(8)

which, we follow from [1] and with (H.2) and weak comparison principle and regularity estimates, implies

$$|v|_1 \leq C_3(\mu |g(u)|_0 + m |\psi_q(u)|_0) \leq C_4 |u|_0^{q-1}(\lambda \delta + m)$$

$$\leq C_3 C_2^{q-1}(\mu \delta + m)(\lambda \delta + m)^{p-1} \frac{1}{p+\rho}$$

$$\equiv M_4((\lambda \delta + m)^{q-1}(\mu \delta + m))^\frac{1}{p+\rho}$$

and

$$v(x) \leq M_4((\lambda \delta + m)^{q-1}(\mu \delta + m))^\frac{1}{p+\rho} d(x, \partial \Omega)$$

(9)

follows from the mean value theorem. The upper estimate for u follows in the same manner.

Next, let $x_0 \in \Omega$ and $R > 0$ be such that $B \equiv B(x_0, R) \subset \Omega$. Here $B(x_0, R)$ denotes the open ball centered at $x_0$ with radius R. Then $(u, v)$ is a supersolution for

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be the solution of

$$
\begin{aligned}
-\Delta \pi &= \lambda f(\pi) + m\psi_p(\pi) \quad \text{in } B \\
-\Delta \pi &= \mu g(\pi) + m\psi_q(\pi) \quad \text{in } B \\
\pi &= \pi = 0 \quad \text{on } \partial B
\end{aligned}
$$

(10)

We shall construct a positive subsolution \((u_0, v_0)\) for (10) with \(u_0 \leq u\) and \(v_0 \leq v\). To this end, let \(\epsilon > 0\) and let \(\tilde{u}, \tilde{v}\) be the solution of

$$
\begin{aligned}
-\Delta \tilde{u} &= \epsilon \tilde{v}^{p-1} + m\psi_p(\tilde{v}) \quad \text{in } B \\
-\Delta \tilde{v} &= \epsilon \tilde{u}^{q-1} + m\psi_q(\tilde{u}) \quad \text{in } B \\
\tilde{u} &= \tilde{v} = 0 \quad \text{on } \partial B
\end{aligned}
$$

(11)

whose existence follows from [3, 5]. Define \(u_0 = \epsilon \tilde{u}, v_0 = \mu \beta \epsilon \tilde{v}\), where \(\beta\) is given by \((H.2)\). A direct calculation gives

$$
\begin{aligned}
\Delta u_0 &= \epsilon \tilde{u}^{p-1}(\epsilon \tilde{v}^{p-1} - m\psi_p(\tilde{v}))
\geq -\lambda \beta(\mu \beta \epsilon \tilde{v}^{p-1} - m(\mu \beta \epsilon \tilde{v})^{p-1}) \\
&\geq -\lambda f(v_0) - m\psi_p(v_0).
\end{aligned}
$$

If \(\|(\lambda \delta + m)(\mu \delta + m)^{p-1}\| > 1, \epsilon\) sufficiently small, and

$$
\begin{aligned}
\Delta v_0 &= \mu \beta \epsilon \tilde{u}^{q-1}(\epsilon \tilde{v}^{q-1} - m\psi_q(\tilde{v}))
\geq -\mu g(u_0) - m\psi_q(u_0)
\end{aligned}
$$

i.e., \((u_0, v_0)\) is a subsolution for (10). Clearly \(u_0 \leq u\) and \(v_0 \leq v\) in B for small \(\epsilon\). Hence there exists a solution \((\pi, \overline{\pi})\) to (10) with \(\overline{\pi} \leq u, \pi \leq v\). Since \(\pi\) is radially symmetric, it follows from [5, Lemma 4] that

$$
u(x) \geq \tilde{M}_1((\lambda \delta + m)(\mu \delta + m)^{p-1})^{\frac{1}{p+q-p-q}} \quad \text{for } |x - x_0| \leq \frac{R}{2}
$$

(12)

for \(\min((\lambda \delta + m)(\mu \delta + m)^{p-1}, (\lambda \delta + m)^{q-1}(\mu \delta + m))\) large, where \(\tilde{M}_1\) is a positive constant independent of \(u, v, \lambda, \mu\).

Let \(\tilde{\Omega} = \Omega \setminus B(x_0, \frac{R}{2})\) and let \(\psi\) be the solution of

$$
\begin{aligned}
\Delta \phi &= 0 \quad \text{in } \tilde{\Omega}, \\
\phi &= 0 \quad \text{on } \partial \tilde{\Omega}, \\
\phi &= 1 \quad \text{on } \partial B(x_0, \frac{R}{2})
\end{aligned}
$$

(13)

Since \(\Delta u \leq 0\) in \(\Omega\), the maximum principle (see, e.g., [8, 10]) implies

$$
u(x) \geq (\tilde{M}_1((\lambda \delta + m)(\mu \delta + m)^{p-1})^{\frac{1}{p+q-p-q}} \phi(x)
\geq \tilde{M}_1((\lambda \delta + m)(\mu \delta + m)^{p-1})^{\frac{1}{p+q-p-q}} d(x, \partial \Omega)
\quad \text{in } \tilde{\Omega},
$$

where \(\tilde{M}_1\) is a positive constant satisfying \(\tilde{M}_1 \phi(x) \geq M_1 d(x, \partial \Omega)\) for \(x \in \tilde{\Omega}\). Combine this and (12), we obtain the lower estimate for \(u\). This completes the proof of Lemma 1. □

Lemma 3 Let \((u, v)\) be a solution to (1) and satisfy

$$
\begin{aligned}
-\Delta w_0 &= \mu g(u) + m\psi_q(u) \quad \text{in } \Omega, \\
w_0 &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
$$

(14)

then for \(\min((\lambda \delta + m)(\mu \delta + m)^{p-1}, (\lambda \delta + m)^{q-1}(\mu \delta + m))\) large, there exist a positive number \(c\) independent of \(u, v, \lambda, \mu\), such that

$$
w_0(x) \geq cd(x, \partial \Omega) \quad \text{for } x \in \Omega.
$$
Proof. Let $\epsilon_0 > 0$. It follows from (H.2) and lemma 1 that for \( \min((\lambda\delta + m)(\mu\delta + m)^{p-1}, (\lambda\delta + m)^{q-1}(\mu\delta + m)) \) large,

\[
\mu g(u(x)) + m\psi_q(u(x)) \geq \mu\beta(u(x))^{q-1} + m\psi_q(u(x)) \\
\geq (\mu\beta + m)[M_1((\lambda\delta + m)(\mu\delta + m)^{p-1})]^{q-1}d(x, \partial\Omega)^{q-1} > 1
\]

if \( d(x, \partial\Omega) > \epsilon_0 \). Thus

\[
\Delta w_0 \leq \begin{cases} 
-1 & \text{if } d(x, \partial\Omega) > \epsilon_0, \\
0 & \text{if } d(x, \partial\Omega) \leq \epsilon_0,
\end{cases}
\]

(15)

and the Lemma follows from the maximum principle and weak comparison principle.

■

Proof of Theorem 1. The existence part of follows from [3]. Let \((u, v)\) and \((u_1, v_1)\) be solutions (1) and suppose that \( \min((\lambda\delta + m)(\mu\delta + m)^{p-1}, (\lambda\delta + m)^{q-1}(\mu\delta + m)) \) is large enough, so Lemma 1 and Lemma 2 can be applied. By Lemma 1,

\[
\frac{M_1}{M_2} u_1 \leq u \leq \frac{M_2}{M_1} u_1 \text{ in } \Omega.
\]

Let \( \alpha = \sup\{c > o : u \geq cu_1 \text{ in } \Omega\} \). Then \( \alpha_0 < \alpha \leq \alpha_0^{-1} \), where \( \alpha_0 = \frac{M_2}{M_1} \). We claim that \( \alpha \geq 1 \). Suppose to the contrary that \( \alpha < 1 \). Let \( q_1, q_2, p_1, p_2 \) be such that \( q_2 > q_1 > q, p_2 > p_1 > p \) and \( p, q > 1 \). Let \( A > 0 \) be such that \( \frac{\delta(u)}{x^{a_0}} \) is nonincreasing for \( x > A \) and define \( \Omega_1 = \{x \in \Omega : u_1(x) > \frac{A}{\alpha_0}\} \). Then

\[
g(\alpha u_1(x)) > \alpha g(u_1(x)) \quad x \in \Omega_1,
\]

while if \( x \in \Omega \setminus \Omega_1 \),

\[
|(g(u_1(x)) + m\psi_q(u_1)) - (g(\alpha u_1(x)) + m\psi_q(\alpha u_1))| \leq K(1 - \alpha),
\]

where \( K = \frac{1}{\alpha_0} \sup\{|x^{a_0} + mx^{a_0-1}| : 0 < x \leq \frac{A}{\alpha_0}\} \), which implies

\[
g(u_1(x)) + m\psi_q(u_1) \geq g(\alpha u_1(x)) + m\psi_q(\alpha u_1) - K(1 - \alpha) \quad \text{for } x \in \Omega \setminus \Omega_1.
\]

Define the operator \( T : C(\overline{\Omega}) \rightarrow C(\overline{\Omega}) \) by \( T(z) = w \) if

\[
\Delta w = -z \quad \text{in } \Omega_1, \quad w = 0 \quad \text{on } \partial\Omega_1.
\]

Let \( w = T(g(\alpha u_1) + m\psi_q(\alpha u_1)) \). Then it follows from (16),(17) and maximum principle that \( w \geq \underline{\omega} \), where \( \underline{\omega} \) satisfies

\[
-\Delta \underline{\omega} = \begin{cases} 
\alpha g_1(\mu g(u_1) + m\psi_q(u_1)) & \text{in } \Omega_1, \\
\mu g(u_1) + m\psi_q(u_1) - K(1 - \alpha) & \text{on } \partial\Omega_1.
\end{cases}
\]

(18)

Let \( w_0 = T(\mu g(u_1) + m\psi_q(u_1)) \). Then \( -\Delta w_0 = \mu g(u_1) + m\psi_q(u_1) \) and therefore

\[
\Delta(\underline{\omega} - \alpha g_1 w_0) = \begin{cases} 
0 & \text{in } \Omega_1, \\
(\alpha g_1 - 1)(\mu g(u_1) + m\psi_q(u_1)) + K(1 - \alpha) & \text{in } \Omega \setminus \Omega_1.
\end{cases}
\]

(19)

Note that there exists a positive constant \( K_1 \) depending only on \( A, \alpha_0, K, q_1 \), such that

\[
|1(\alpha g_1 - 1)(\mu g(u_1) + m\psi_q(u_1)) + K(1 - \alpha)| \leq K_1(1 - \alpha) \quad \text{in } \Omega \setminus \Omega_1.
\]

Using regularly estimates, we obtain for \( r > N \),

\[
|\underline{\omega} - \alpha g_1 w_0|_1 \leq CK_1(1 - \alpha)(\int_{\Omega_1} dx)^{\frac{1}{r}}.
\]

(20)
Since
\[ M_1((\lambda \delta + m)(\mu \delta + m)^{p-1}) \frac{1}{\alpha_0} d(x, \partial \Omega) \leq u_1(x) \leq \frac{A}{\alpha_0} \quad \text{on} \; \Omega \setminus \Omega_1 \]
it follows that
\[ d(x, \partial \Omega) \leq \frac{A}{\alpha_0 M_1((\lambda \delta + m)(\mu \delta + m)^{p-1})} \quad \text{for} \; x \in \Omega \setminus \Omega_1 \]
and therefore the right-hand side of (20) goes to 0 as \((\lambda \delta + m)(\mu \delta + m)^{p-1} \to \infty\). Let \(\epsilon > 0\), then it follows from (20) and the mean value theorem and weak comparison principle that
\[
\bar{w}(x) - \alpha^{q_1} w_0 \geq -\epsilon(1 - \alpha)d(x, \partial \Omega), \quad x \in \Omega,
\]
for
\[
\min((\lambda \delta + m)(\mu \delta + m)^{p-1}, (\lambda \delta + m)^{p-1}(\mu \delta + m))
\]
large, which implies by Lemma 2 that
\[
\bar{w}(x) - \alpha^{q_2} w_0(x) \geq (\alpha^{q_1} - \alpha^{q_2})w_0(x) - \epsilon(1 - \alpha)d(x, \partial \Omega)
\]
\[
\geq c\alpha^{q_1}(1 - \alpha^{q_2-p_1})d(x, \partial \Omega) - \epsilon(1 - \alpha)d(x, \partial \Omega)
\]
\[
\geq [\min(1, q_2 - q_1)\alpha^{q_1} - \epsilon](1 - \alpha)d(x, \partial \Omega) > 0
\]
if \(\epsilon\) sufficiently small. Consequently, \(w(x) \geq \bar{w}(x) \geq \alpha^{q_2} w_0(x)\), or
\[
T(\mu g(\alpha u_1) + m \psi_q(\alpha u_1)) \geq \alpha^{q_2} T(\mu g(u_1) + m \psi_q(u_1)). \tag{21}
\]
Since
\[
\Delta v = -\mu g(u) - m \psi_q(u) \leq -\mu g(\alpha u_1) - m \psi_q(\alpha u_1),
\]
it follows from (21) that
\[
v \geq T(\mu g(\alpha u_1) + m \psi_q(\alpha u_1)) \geq \alpha^{q_2} T(\mu g(u_1) + m \psi_q(u_1)) = \alpha^{q_2} v_1.
\]
This implies
\[
\Delta u = -\lambda f(v) - m \psi_p(v) \leq -\lambda f(\alpha^{q_2} v_1) - m \psi_p(\alpha^{q_2} v_1). \tag{22}
\]
Now similarly
\[
T(\lambda f(\alpha^{q_2} v_1) + m \psi_p(\alpha^{q_2} v)) \geq \alpha^{p_2 q_2} T(\lambda f(v_1) + m \psi_p(v_1)). \tag{23}
\]
(22),(23) and the maximum principle imply
\[
u \geq T(\lambda f(\alpha^{q_2} v_1) + m \psi_p(\alpha^{q_2} v)) \geq \alpha^{p_2 q_2} T(\lambda f(v_1) + m \psi_p(v_1)) = \alpha^{p_2 q_2} u_1,
\]
which is a contradiction since \(\alpha^{p_2 q_2} > \alpha\). Thus \(\alpha \geq 1\), i.e., \(u \geq u_1\) and therefore \(u = u_1\).

Similarly, \(v = v_1\), completing the proof of Theorem 1. \(\square\)

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