New Application of Weierstrass Function Methods to Two Types of Boussinesq Equations

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Abstract: For the two generalized forms of the well-known Boussinesq equation, we apply the theory of Weierstrass function method that has been applied by Kuru and so on. We get some traveling wave solutions that never seen before. The method is direct and concise, and free of the tedious calculation. The traveling wave solutions are expressed by the hyperbolic functions and confirmed to be solitary wave solutions.

Keywords: Weierstrass function; generalized Boussinesq equations; traveling wave solutions

1 Introduction

The nonlinear phenomena exist in all the fields including either the scientific work or engineering fields. Many nonlinear evolution equations are playing important role in the analysis of some phenomena.

In order to obtain the travelling wave solutions to these nonlinear evolution equations, many methods were attempted, such as Hirota’s bilinear transformation, the tanh-sech method, G'/G-expansion method [1], Jacobian elliptic functions method[2] and polynomial expansion method[3]. The above methods derived many types of solutions from most nonlinear evolution equations.

In recent years, in order to find the exact solutions of some travelling wave equations, Kuru[4,5] discussed the BBM-like equation, and Estèvez, Kuru [6]analyzed another type of generalized BBM equations. When they searched for the solutions by the factorization technique, they paid much attention to ordinary differential equation:

\[
\left(\frac{d\varphi}{d\theta}\right)^2 = P_4(\varphi) = a_0\varphi^4 + 4a_1\varphi^3 + 6a_2\varphi^2 + 4a_3\varphi + a_4,
\]

with the help of Weierstrass function \( \wp(\theta; g_2, g_3) \), they obtained many solutions of the two equations.

For the differential equation (1), only if the power of \( \varphi \) are the integer numbers between 0 and 4, can we guarantee the integrability of (1). Therefore we may consider the following possible cases.

For simplicity, we introduce two invariants

\[
g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2
\]
\[
g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_2^2 - a_1^2a_4
\]

and a discriminant \( \Delta = g_3^3 - 27g_2^2 \), then the solutions of differential equation(1) have the following form:

\[
\varphi(\theta) = \varphi_0 + \frac{1}{4}P_\varphi(\varphi_0)(\varphi(\theta; g_2, g_3) - \frac{1}{24}P_{\varphi\varphi}(\varphi_0))^{-1},
\]

where \( \varphi_0 \) is one of the roots of the polynomial \( P_4(\varphi) \), and \( P_\varphi(\varphi_0) \), \( P_{\varphi\varphi}(\varphi_0) \) respectively denote the first and second derivative of \( P_4(\varphi) \) with respect to \( \varphi \) at \( \varphi_0 \). Particularly, if \( \Delta = g_3^3 - 27g_2^2 = 0 \), then the Weierstrass function \( \varphi(\theta; g_2, g_3) \) satisfies these conditions:

\[
\varphi(\theta; 12b^2, -8b^3) = b + 3b \sinh^{-2}(\sqrt{3}b\theta)
\]
\[ \varphi(\theta; 12b^2, 8b^3) = -b + 3b\sin^{-2}(\sqrt{3}b) \] (5)

Once we get the solution of (1) with the form of (3), we will get the solutions of many partial differential equations.

The present work is interested in two equations: the dispersive Boussinesq equation (B(n,n) equation):

\[ u_{tt} - u_{xx} - (u^n)_{xx} + (u^n)_{xxxx} = 0, \quad n \geq 1 \] (6)

the generalized Boussinesq-like equation (B(2n,2n) equation)

\[ u_{tt} - (u^{2n})_{xx} + (u^{2n})_{xxxx} = 0, \quad n \geq 1. \] (7)

Both the equations were researched in recent years. The dispersive Boussinesq equation (6) was studied by Zhenya Yan and George Bluman [7] with four direct ansatze and some compactons were obtained. Mustafa Inc [8] applied new extended sn-cn method to (6) and solitary wave, periodic wave solutions and Jacobi elliptic function solutions were found. The generalized Boussinesq-like equation (7) with fully nonlinear dispersion was analysed by Yonggui Zhu [9] with decomposition method. Lijun Zhang, Li-Qun Chen, et al [10] investigated (7) and got abundant compactons, peakons and solitary solutions.

The organization of this paper is as follows: In Section 2 and 3, we applied the Weierstrass function method to the dispersive Boussinesq equation (B(n,n) equation) (6) and the generalized Boussinesq-like equation (B(2n,2n) equation) (7) separately. At last, we give some conclusions.

2 The Nonlinearly Dispersive Boussinesq Equations

Consider the nonlinear differential equation

\[ u_{tt} - u_{xx} - (u^n)_{xx} + (u^n)_{xxxx} = 0 \] (8)

Let \( u^n(x, t) = W(\xi) \), \( \xi = x - ct \), integrate once and let \( D \) the integration constant, we get that

\[ (W')^2 = \frac{2n(1-c^2)}{n+1}W^{\frac{n}{2}+1} + W^2 + DW \] (9)

Let \( W = \varphi^p \), then

\[ (\varphi')^2 = \frac{2n(1-c^2)}{(n+1)p^2}\varphi^{\left(\frac{1}{p} - 1\right)p+2} + \frac{1}{p^2}\varphi^2 + \frac{D}{p^3}\varphi^{2-p} \] (10)

If \( D = 0 \), then \( p \in \{ \pm \frac{2n}{n+1}, \pm \frac{n}{n+1} \} \).

Case 1 if \( p = \frac{2n}{n+1} \), then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = \frac{(1-c^2)(n-1)^2}{2n(n+1)} + \frac{(n-1)^2}{4n^2}\varphi^2. \] (11)

The polynomial \( P(\varphi) \) has two roots:

\[ g_2 = \frac{(n-1)^4}{192n^4}, \quad g_3 = -\frac{(n-1)^6}{13824n^6}, \quad \Delta = 0. \]

Therefore

\[ \varphi(\xi; g_2, g_3) = \left( \frac{n-1}{4n} \right)^2 \left( \frac{1}{3} + \sinh^{-2} \left( \frac{n-1}{4n} \xi \right) \right) \] (12)

and

\[ \varphi(\xi; g_2, g_3) = \left( \frac{n-1}{4n} \right)^2 \left( \frac{1}{3} - \sinh^{-2} \left( \frac{n-1}{4n} \xi \right) \right) \]

It is obviously that the solution of the ordinary differential equation (11) is

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\( \varphi(\xi) = \pm \sqrt{\frac{2n(c^2 - 1)}{n + 1}} \left( 2 \cosh^2 \left( \frac{n-1}{4n} \xi \right) - 1 \right), \) \hfill (13)

Therefore the solution of partial differential equation (8) is

\[ u(x,t) = \left( \pm \sqrt{\frac{2n(c^2 - 1)}{n + 1}} \left( 2 \cosh^2 \left( \frac{n-1}{4n} (x-ct) \right) - 1 \right) \right)^{\frac{2}{n^2}}. \] \hfill (14)

Case 2 if \( p = \frac{-2n}{n^2 - 1}, \) then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = \frac{(1 - c^2)(n-1)^2}{2n(n+1)} \varphi^4 + \frac{(n-1)^2}{4n^2} \varphi^2. \] \hfill (15)

The polynomial \( P(\varphi) \) has four roots: \( \varphi = \pm \sqrt{\frac{n + 1}{2n(c^2 - 1)}} \) and \( \varphi = 0 \)

\[ g_2 = \frac{(n-1)^4}{12n^4}, \quad g_3 = -\frac{(n-1)^6}{216n^6}, \quad \Delta = 0. \] \hfill (16)

Due to the same discriminants with Case 1, there exist the same Weierstrass functions (12). It is obviously that the solution of the ordinary differential equation (15) is

\[ \varphi(\xi) = \pm \sqrt{\frac{n + 1}{2n(c^2 - 1)}} \left( 2 \cosh^2 \left( \frac{n-1}{4n} \xi \right) - 1 \right)^{-1}, \] \hfill (17)

Therefore the solution is same to (14) that obtained in Case 1.

Case 3 if \( p = \frac{n}{n^2 - 1}, \) then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = \frac{2(1 - c^2)(n-1)^2}{n(n+1)} \varphi + \frac{(n-1)^2}{n^2} \varphi^2. \] \hfill (18)

The polynomial \( P(\varphi) \) has two roots: \( \varphi = \frac{2n(c^2 - 1)}{n+1} \) and \( \varphi = 0 \)

\[ g_2 = \frac{(n-1)^4}{12n^4}, \quad g_3 = -\frac{(n-1)^6}{216n^6}, \quad \Delta = 0. \] \hfill (16)

Then,

\[ \varphi(\xi; g_2; g_3) = \left( \frac{n-1}{2n} \right)^2 \left( \frac{1}{3} + \sinh^{-2} \left( \frac{n-1}{2n} \xi \right) \right) \] \hfill (19)

and

\[ \varphi(\xi; g_2; g_3) = \left( \frac{n-1}{2n} \right)^2 \left( \frac{1}{3} - \sinh^{-2} \left( \frac{n-1}{2n} \xi \right) \right) \]

It is obviously that the solution of the ordinary differential equation (18) is

\[ \varphi(\xi) = \frac{2n(c^2 - 1)}{n + 1} \cosh^2 \left( \frac{n-1}{n} \xi \right), \] \hfill (20)

Therefore the solution of partial differential equation (8) is

\[ u(x,t) = \left( \frac{2n(c^2 - 1)}{n + 1} \cosh^2 \left( \frac{n-1}{n} (x-ct) \right) \right)^{\frac{1}{n^2}}. \] \hfill (21)

Case 4 if \( p = -\frac{n}{n^2 - 1}, \) then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = \frac{2(1 - c^2)(n-1)^2}{n(n+1)} \varphi^3 + \frac{(n-1)^2}{n^2} \varphi^2. \] \hfill (22)

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The polynomial $P(\varphi)$ has two roots: $\varphi = \frac{n+1}{2n(c^2-1)}$ and $\varphi = 0$

$$g_2 = \frac{(n-1)^4}{12n^4}, \quad g_3 = -\frac{(n-1)^6}{216n^6}, \quad \Delta = 0$$

(23)

Due to the same discriminants with Case 3, there exist the same Weierstrass functions (19). It is obviously that the solution of the ordinary differential equation (22) is

$$\varphi(\xi) = \frac{n+1}{2n(c^2-1)} \cosh^{-2}\left(\frac{n-1}{n} \xi\right),$$

(24)

Therefore the solution of partial differential equation is

$$u(x,t) = \left(\frac{n+1}{2n(c^2-1)} \cosh^{-2}\left(\frac{n-1}{n} (x-ct)\right)\right)^{\frac{1}{\Delta}}.$$  

(25)

Obviously the result is same to (21). Compare to the solutions that were shown in[7,8], these two results haven't been given so far. The solutions are solitary wave solutions.

3 The Generalized Boussinesq-like Equation

Consider the nonlinear differential equation

$$u_{tt} - (u^{2n})_{xx} + (u^{2n})_{xxxx} = 0, \quad n \geq 1$$

(26)

Let $u^n(x,t) = W(\xi)$, $\xi = x - ct$, integrate once and let $D$ the integration constant, we get that

$$(W')^2 = -\frac{4nc^2}{2n+1} W^{2n+1} + W^2 + DW$$

(27)

Let $W = \varphi^p$, then

$$(\varphi')^2 = -\frac{4nc^2}{(2n+1)p^2} \varphi^{(\frac{1}{p})^2 + 1} + \frac{1}{p^2} \varphi^2 + \frac{D}{p^2} \varphi^{2-p}$$

(28)

If $D = 0$, then $p \in \{\pm \frac{4n}{2n+1}, \pm \frac{2n}{2n+1}\}$

Case 1 if $p = \frac{4n}{2n+1}$, then

$$\left(\frac{d\varphi}{d\xi}\right)^2 = -\frac{c^2(2n-1)^2}{4n(2n+1)} + \frac{(2n-1)^2}{16n^2} \varphi^2.$$  

(29)

The polynomial $P(\varphi)$ has two roots: $\varphi = \pm \sqrt{\frac{n}{2n+1} c}$

$$g_2 = \frac{(2n-1)^4}{3072n^4}, \quad g_3 = -\frac{(2n-1)^6}{884736n^6}, \Delta = 0.$$  

Then

$$\varphi(\xi; g_2, g_3) = \left(\frac{2n-1}{8n}\right)^2 \left(\frac{1}{3} + \sinh^{-2}\left(\frac{2n-1}{8n} \xi\right)\right)$$

(30)

and

$$\varphi(\xi; g_2, g_3) = \left(\frac{2n-1}{8n}\right)^2 \left(\frac{1}{3} - \sinh^{-2}\left(\frac{2n-1}{8n} \xi\right)\right)$$

(31)

It is obviously that the solution of the ordinary differential equation(29) is

$$\varphi(\xi) = \pm 2c \sqrt{\frac{n}{2n+1}} \left(2 \cosh^2\left(\frac{2n-1}{8n} \xi\right) - 1\right),$$

(31)

Therefore the solution of partial differential equation (26) is

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\( u(x, t) = \left( \pm 2c \sqrt{\frac{n}{2n+1}} \left( 2 \cosh^2 \left( \frac{2n-1}{8n} (x - ct) \right) - 1 \right) \right)^{\frac{1}{n+1}}. \)  

(32)

Case 2 if \( p = -\frac{4n}{2n-1} \), then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = -c^2 \left( 2n - 1 \right)^2 \varphi^4 + \frac{(2n - 1)^2}{16n^2} \varphi^2. \]

(33)

The polynomial \( P_{\varphi} \) has three roots: \( \varphi = \pm \sqrt{\frac{2n+1}{4n^2}} \) and \( \varphi = 0 \)

\[ g_2 = \frac{(2n - 1)^4}{3072n^4}, \quad g_3 = -\frac{(2n - 1)^6}{884736n^6}, \Delta = 0. \]

(34)

Due to the same discriminants with Case 1, there exist the same Weierstrass functions (30). It is obviously that the solution of the ordinary differential equation (33) is

\[ \varphi(\xi) = \pm \frac{1}{2c} \sqrt{\frac{2n+1}{n}} \left( 2 \cosh^2 \left( \frac{2n-1}{8n} \xi \right) - 1 \right)^{-1}, \]

(35)

Therefore the solution of partial differential equation is same to that of case 1.

Case 3 if \( p = \frac{2n}{2n-1} \), then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = -c^2 \left( 2n - 1 \right)^2 \varphi^4 + \frac{(2n - 1)^2}{4n^2} \varphi^2. \]

(36)

The polynomial \( P_{\varphi} \) has two roots: \( \varphi = \frac{4n^2}{2n+1} \) and \( \varphi = 0 \)

\[ g_2 = \frac{(2n - 1)^4}{192n^4}, \quad g_3 = -\frac{(2n - 1)^6}{13824n^6}, \Delta = 0. \]

(37)

Therefore,

\[ \varphi(\xi; g_2, g_3) = \left( \frac{2n-1}{4n} \right)^2 \left( \frac{1}{3} + \sinh^{-2} \left( \frac{2n-1}{4n} \xi \right) \right) \]

(37)

and

\[ \varphi(\xi; g_2, g_3) = \left( \frac{2n-1}{4n} \right)^2 \left( \frac{1}{3} - \sinh^{-2} \left( \frac{2n-1}{4n} \xi \right) \right) \]

(37)

It is obviously that the solution of the ordinary differential equation (36) is

\[ \varphi(\xi) = \frac{4n^2}{2n+1} \cosh^2 \left( \frac{2n-1}{4n} \xi \right), \]

(38)

Therefore the solution of partial differential equation (26) is

\[ u(x, t) = \left( \frac{4n^2}{2n+1} \cosh^2 \left( \frac{2n-1}{4n} (x - ct) \right) \right)^{\frac{1}{n+1}}. \]

(39)

Case 4 if \( p = -\frac{2n}{2n-1} \), then

\[ \left( \frac{d\varphi}{d\xi} \right)^2 = -c^2 \left( 2n - 1 \right)^2 \varphi^3 + \frac{(2n - 1)^2}{4n^2} \varphi^2. \]

(40)

The polynomial \( P_{\varphi} \) has roots: \( \varphi = \frac{2n+1}{4n^2} \) and \( \varphi = 0 \)

\[ g_2 = \frac{(2n - 1)^4}{192n^4}, \quad g_3 = -\frac{(2n - 1)^6}{13824n^6}, \Delta = 0. \]

(41)

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Due to the same discriminants with Case 1, there exist the same Weierstrass functions (37). It is obviously that the solution of the ordinary differential equation (40) is

\[ \varphi(\xi) = \frac{2n + 1}{4nc^2} \cosh^{-2}\left(\frac{2n - 1}{4n} \xi\right), \]

Therefore the solution of partial differential equation is the same to (39). Compare to the solutions that were shown in [6,7], these two results haven’t been given so far. The solutions are solitary wave solutions.

4 Conclusions

In the paper, for the two interesting generalized forms of the well-known Boussinesq equation, we apply the Weierstrass function method and obtain some new solitary wave solutions, and these solutions have many fluent properties. The method is powerful for some evolutionary equations with arbitrary order nonlinearity. We will make more attempts to verify the strength of the method.

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