Solution of an Initial Boundary Value Problem for Non-Planar Burgers Equation Using Hermite Interpolation

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Abstract: We study an initial boundary value problem for the non-planar Burgers equation using Hermite interpolants. Numerical solution and solutions obtained by Hermite interpolation are compared and are found to be in good agreement.

Keywords: Hermite interpolation; Non-planar Burgers equation; Initial boundary value problems

1 Introduction

In this paper, we study an initial boundary value problem (IBVP) for the non-planar Burgers equation, namely,

\[ u_t + u^\alpha u_x + \frac{j u}{2(t + 1)} = \epsilon u_{xx}, \quad 0 < x < 1, \quad t > 0, \]  
\[ u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \]  
\[ u(0, t) = a, \quad u(1, t) = b, \quad t \geq 0, \]  

where \( \epsilon > 0 \) is small, \( \alpha \geq 1 \) is an integer and \( j > 0 \) are parameters. Further, \( a \) and \( b \) are non-negative constants. We assume throughout that \( u_0(x) \geq 0 \) is sufficiently smooth on \([0, 1]\). Equation (1) has applications in nonlinear acoustics (see, for example, Enflo and Hedberg [2]). Following Grundy [3], we construct Hermite interpolants to approximate the solution of the IBVP (1)–(3). Then, we compare the Hermite interpolants with a numerical solution of the IBVP (1)–(3) obtained by a finite difference scheme due to Dawson [1]. Based on excellent agreement between the Hermite interpolants and numerical solution, we may conclude that a suitable Hermite interpolant solution is a good approximation to the solution of the IBVP (1)–(3) for all time.

Now, we define Hermite interpolants for a function (see, for example, Grundy [3]). Let \( f(x) \) be a sufficiently smooth function defined on \([0, 1]\). Further, let \( f^{(r)}(x) \) be the \( r \)th derivative of \( f(x) \); \( f^{(r)}(0) \) and \( f^{(r)}(1) \) are known for \( r = 0, 1, ..., n \) for some positive integer \( n \). Then, \( n \)th order Hermite interpolant of \( f(x) \), denoted by \( p_n(x) \), is written as

\[ p_n(x) = \sum_{r=0}^{n} \left\{ f^{(r)}(0)Q^n_r(x) + (-1)^r f^{(r)}(1)Q^n_r(1 - x) \right\}, \quad x \in [0, 1], \]  

where \( Q^n_r(x) \) is a polynomial of degree \( 2n + 1 \) on \([0, 1]\), and is given by

\[ Q^n_r(x) = \frac{x^r}{r!} (1 - x)^{n+1} \sum_{s=0}^{n-r} \frac{(n + s)!}{s! n!} x^s. \]  

Thus, a Hermite interpolant approximates a function over an interval by making use of the values of the function and a certain number of its derivatives at the end points of the interval. The error in approximating the function \( f \) by the Hermite interpolant \( p_n \) on \([0, 1]\) is given by

\[ f(x) - p_n(x) = (-1)^n x^{n+1} (1 - x)^{n+1} f^{(2n+2)}(\xi)/(2n + 2)!, \]  

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for some $\xi \in (0, 1)$ and $f^{(2n+2)}$ is assumed to be continuous in $(0, 1)$.

The Hermite interpolation polynomial given by (4)–(5) is discussed in detail by Phillips [8]. Grundy and Phillips [6] exemplified the application of Hermite interpolants to estimate the initial values for solving a boundary value problem posed for ODEs of the form $y''(x) = f(x, y)$ on $[0, 1]$. We may refer to Lanczos [7] and Grundy ([4], [5]) for a related study.

Sachdev and his collaborators (see [9], [10]) have studied the large time behaviour of periodic solutions of some generalised Burgers equations, including the non-planar Burgers equation, using a perturbative technique.

In the next section, we approximate the solution of the IBVP (1)–(3) by a suitable Hermite interpolant and then present a comparison of the Hermite interpolants with the numerical solution of the IBVP (1)–(3). Finally section 3 presents the conclusions.

2 Hermite interpolant solution of IBVP (1)–(3)

In this section, we find Hermite interpolants $p_2, p_3, p_4$ for approximating the solution of the initial boundary value problem (1)–(3) and compare them with numerical solution of the IBVP (1)–(3). This approximation is valid for all time $t$. The accuracy of the approximation depends on the order of the Hermite interpolant and the compatibility of the initial and boundary conditions with the given PDE.

Rewrite (1) as

$$u_{xx} = \frac{1}{\epsilon} \left[ u_t + u^\alpha u_x + \frac{j u}{2(t + 1)} \right].$$

We make use of the Green’s function for the operator $\frac{\partial^2}{\partial x^2}$ subject to homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$ to arrive at the following integro-differential equation

$$u(x, t) = a + (b - a)x + \frac{1}{\epsilon} \int_0^1 K(x, s) \left( u_t + u^\alpha u_s + \frac{j u}{2(t + 1)} \right) ds, \tag{1}$$

where $K(x, s) = \begin{cases} s(x - 1), & 0 \leq s \leq x \\ x(s - 1), & x \leq s \leq 1 \end{cases}$. The first two terms on the right hand side of (1) are due to the nonhomogeneous boundary conditions (see (3)) at $x = 0$ and $x = 1$. Further, $u_t$ and $u_s$ in (1) are partial derivatives of $u(s, t)$ with respect to $t$ and $s$ respectively. Performing an integration by parts in (1), we arrive at

$$u(x, t) = a + (b - a)x + \frac{1}{\epsilon} \int_0^1 K(x, s) \left( u_t + \frac{j u}{2(t + 1)} \right) ds$$

$$- \int_0^x (x - 1) u_{\alpha + 1} ds - \int_x^1 (x - 1) u_{\alpha + 1} ds. \tag{2}$$

Differentiating (2) with respect to $x$, we get

$$u_x(x, t) = \frac{1}{\epsilon} \left[ \int_0^x \left( u_t + \frac{j u}{2(t + 1)} \right) ds + \int_x^1 \left( u_t + \frac{j u}{2(t + 1)} \right) ds \right.$$

$$- \int_0^x u_{\alpha + 1} ds - \int_x^1 u_{\alpha + 1} ds \bigg] + (b - a). \tag{3}$$

Let

$$u_x(0, t) = V_0(t) \quad \text{and} \quad u_x(1, t) = V_1(t). \tag{4}$$

Following (4) and (5), we write the Hermite interpolant approximation to $u(x, t)$ as

$$p_n(x, t) = \sum_{r=0}^n \left\{ u_{x, r}(0, t)Q_r^\alpha(x) + (-1)^r u_{x, r}(1, t)Q_r^\alpha(1 - x) \right\}, \quad x \in [0, 1], \tag{5}$$

where $Q_r^\alpha(x)$ is as in (5) and $u_{x, r}$ is the $r^{th}$ partial derivative of $u$ with respect to $x$. It may be noted here that $u_{x, r}(0, t)$ and $u_{x, r}(1, t)$ can be expressed in terms of $V_0(t)$, $V_1(t)$ and their derivatives.
We find the Hermite interpolants \( p_2, p_3 \) and \( p_4 \) and compare with numerical solution of the IBVP (1)–(3) for specific initial and boundary conditions. For this purpose, we need to find \( u_{x,r}(0, t) \) and \( u_{x,r}(1, t) \), \( r = 1, 2, 3, 4 \) (see (5)). We give below derivation of expressions for \( u_{x,r}(0, t) \) and \( u_{x,r}(1, t) \) in terms of \( V_0(t) \), \( V_1(t) \) and their derivatives. Clearly from (3) and (4)

\[
\begin{align*}
\ u_{x,0}(0, t) &= u(0, t) = a, \quad u_{x,0}(1, t) = u(1, t) = b, \\
\ u_{x,1}(0, t) &= V_0(t), \quad u_{x,1}(1, t) = V_1(t).
\end{align*}
\]

Making use of the equation (1) and the boundary conditions (3), we get

\[
\ u_{x,2}(0, t) = \frac{1}{\epsilon} \left( a^\alpha V_0 + \frac{\alpha j}{2(t+1)} \right), \quad u_{x,2}(1, t) = \frac{1}{\epsilon} \left( b^\alpha V_1 + \frac{bj}{2(t+1)} \right).
\]

We illustrate the computation of \( u_{x,3}(0, t) \) :

\[
\begin{align*}
\ u_{x,3}(0, t) &= \frac{1}{\epsilon} \left( u_t + u^\alpha u_x + \frac{j u}{2(t+1)} \right) (0, t), \\
&= \frac{1}{\epsilon} \left( u_{tx} + u^\alpha u_{xx} + \alpha \alpha^{-1} u_x^2 + \frac{j u_x}{2(t+1)} \right) (0, t), \\
&= \frac{1}{\epsilon} \left( V_0'' + a^\alpha u_{x,2}(0, t) + \alpha \alpha^{-1} V_0^2 + \frac{j V_0}{2(t+1)} \right).
\end{align*}
\]

Similarly, we can compute the other coefficients in (5) :

\[
\begin{align*}
\ u_{x,3}(1, t) &= \frac{1}{\epsilon} \left( V_1'' + b^\alpha u_{x,2}(1, t) + \alpha \alpha^{-1} V_1^2 + \frac{j V_1}{2(t+1)} \right), \\
\ u_{x,4}(0, t) &= \frac{1}{\epsilon} \left( a^\alpha V_0'' - \frac{j a}{2\epsilon(t+1)^2} + a^\alpha u_{x,3}(0, t) \\
&\quad + \left( 3 \alpha \alpha^{-1} V_0 + \frac{j}{2(t+1)} \right) u_{x,2}(0, t) + \alpha (\alpha - 1) \alpha^{-2} V_0^3 \right), \\
\ u_{x,4}(1, t) &= \frac{1}{\epsilon} \left( b^\alpha V_1'' - \frac{j b}{2\epsilon(t+1)^2} + b^\alpha u_{x,3}(1, t) \\
&\quad + \left( 3 \alpha \alpha^{-1} V_1 + \frac{j}{2(t+1)} \right) u_{x,2}(1, t) + \alpha (\alpha - 1) b^\alpha V_1^3 \right).
\end{align*}
\]

We require that initial and boundary conditions be compatible at \( x = 0 \) and \( x = 1 \), that is,

\[
\ u_{x,r}(0, 0) = u_{0}^{(r)}(0), \quad u_{x,r}(1, 0) = u_{0}^{(r)}(1), \quad r = 0, 1, 2, 3, 4.
\]

Here \( u_{0}^{(r)}(x) \) is the \( r \)th derivative of \( u_0(x) \) and \( u_{x,r}(0, 0) \) and \( u_{x,r}(1, 0) \) are determined above. This results in the following conditions on \( V_0(t) \), \( V_1(t) \) and their first derivatives at \( t = 0 \):

\[
\begin{align*}
V_0(0) &= u_{x,1}(0, 0) = u_0(0), \quad V_1(0) = u_{x,1}(1, 0) = u_0(1), \\
V_0'(0) &= \epsilon u_{x,3}(0, 0) - a^\alpha u_{x,2}(0, 0) - \alpha \alpha^{-1} V_0^2(0) - \frac{j V_0(0)}{2}, \\
V_1'(0) &= \epsilon u_{x,3}(1, 0) - b^\alpha u_{x,2}(1, 0) - \alpha b^\alpha V_1^2(0) - \frac{j V_1(0)}{2}.
\end{align*}
\]

Thus, having determined \( p_n(x, t) \) \( n = 2, 3, 4 \) in terms of the unknown functions \( V_0(t), V_1(t) \) and their derivatives, we replace \( u \) and its first partial derivatives in the right hand side of (3) by \( p_n \) and its corresponding first partial derivatives. Then, letting \( x \to 0+ \) and \( x \to 1- \) in the resulting equation, we obtain a system of ordinary differential equations for \( V_0 \) and \( V_1 \) as

\[
\begin{align*}
V_0 &= b - a + \frac{a^{\alpha+1}}{\epsilon(\alpha + 1)} + \frac{1}{\epsilon} \int_0^1 \left[ (s - 1) \left( \frac{\partial p_n}{\partial t} + \frac{j p_n}{2(t+1)} \right) - \frac{p_n^{\alpha+1}}{\alpha + 1} \right] ds, \\
V_1 &= b - a + \frac{b^{\alpha+1}}{\epsilon(\alpha + 1)} + \frac{1}{\epsilon} \int_0^1 \left[ s \left( \frac{\partial p_n}{\partial t} + \frac{j p_n}{2(t+1)} \right) - \frac{p_n^{\alpha+1}}{\alpha + 1} \right] ds.
\end{align*}
\]
For \( n = 2 \), \( p_n \) involves the values of \( u_{x,r}(x, t) \) for \( r = 0, 1, 2 \) at \( x = 0 \) and \( x = 1 \). Substituting \( p_2(x, t) \) in (9) and simplifying we arrive at a system of two first order nonlinear ordinary differential equations for the unknown functions \( V_0 \) and \( V_1 \). To solve this system of ODEs, we need the initial values \( V_0(0) \) and \( V_1(0) \) given in (8). For a set of parameter values, \( \alpha = 1, j = 1, \epsilon = 0.1 \) and \( a = b = 0 \), we give below the system of ODEs

\[
\begin{align*}
V'_0(t) &= -\left(2.6 + \frac{0.5}{1+t}\right) V_0(t) - 1.6 V_1(t) + 0.40303 V_0(t)V_1(t) \\
&\quad - 0.315152 (V_0^2(t) + V_1^2(t)) \\
V'_1(t) &= -1.6 V_0(t) - \left(2.6 + \frac{0.5}{1+t}\right) V_1(t) + 0.40303 V_0(t)V_1(t) \\
&\quad - 0.315152 (V_0^2(t) + V_1^2(t)).
\end{align*}
\tag{10}
\]

Similarly, when we substitute \( p_3 \) or \( p_4 \) in (9), we arrive at a system of two second order nonlinear ODEs for \( V_0 \) and \( V_1 \). Thus, we need to solve a system of ODEs for \( V_0 \) and \( V_1 \) subject to initial conditions given in (8). This is done numerically. The system of nonlinear ODEs coming from (9) are of the form

\[
\frac{d^2V}{dt^2} = F(t,V,V')
\]

where \( V = (V_0, V_1) \) and the components of \( F \) are polynomials in \( V_0, V_1 \) for \( n = 3, 4 \). So the local existence and uniqueness of solution is guaranteed subject to the initial data (8). It is worthwhile to note that the Hermite interpolant solution \( p_n(x, t) \) approximately solves the IBVP (1)–(3) subject to the initial profile \( p_n(x, 0) \) for all time \( t \).

For our computations, we have chosen \( u_0(x) = \sin(\pi x) \) such that \( a = b = 0 \) (compatibility condition). Figure 1 shows an excellent agreement between the simulated initial profile \( p_4(x, 0) \) with \( \alpha = 1, j = 1 \) and \( \epsilon = 0.1 \) and the initial profile \( u_0(x) = \sin \pi x \); the error is of order \( O(10^{-5}) \). In fact, when \( \alpha = 1, p_4(x, 0) \) does not depend on \( j \) and \( \epsilon \) and is given by

\[
p_4(x,0) = \pi \{ Q_1^1(x) + Q_1^1(1-x) \} - \pi^3 \{ Q_3^1(x) + Q_3^1(1-x) \},
\]

the 4th order Hermite interpolant of \( \sin \pi x \) on \([0, 1]\).

We have compared the numerical solution of the IBVP (1)–(3), obtained by a finite difference scheme due to Dawson [1], and the Hermite interpolants \( p_2(x, t), p_3(x, t) \) and \( p_4(x, t) \) at different times for different values of \( \alpha, j \) and \( \epsilon \). Figures 2 and 3 show the numerical and Hermite interpolant solutions \( p_2(x, t), p_3(x, t), p_4(x, t) \) at times \( t = 1, 5 \) for \( \alpha = 1, j = 1 \) and \( \epsilon = 0.1 \).

At time \( t = 1 \), the maximum absolute error in \( p_4(x, t) \) with respect to the numerical solution is of order \( O(10^{-3}) \), whereas at \( t = 5 \) it is of order \( O(10^{-5}) \). The maximum values of numerical solution of (1)–(3) for \( \alpha = 1, j = 1 \) and \( \epsilon = 0.1 \) at \( t = 1, 5 \) are 0.2325 and 0.0025 respectively. Further, the maximum absolute errors in \( p_2 \) and \( p_3 \), depicted in Figures 2 and 3, with respect to the numerical solution at \( t = 1 \) and \( t = 5 \) are \( O(10^{-2}) \) and \( O(10^{-4}) \) respectively.

![Figure 1](image.png)

Figure 1: Comparison of initial profile \( u_0(x) = \sin \pi x \) and the simulated initial profile \( p_4(x, 0) \) for \( \epsilon = 0.1, \alpha = 1 \) and \( j = 1 \).
For $\alpha = 2$, it is worthwhile to note that the compatibility conditions, as laid down in (7), are not satisfied by $u_{x, A}(0, 0)$ and $u_{x, A}(1, 0)$ with $u_0(x) = \sin \pi x$. Therefore, we compute $p_3(x, t)$ so that, $\sin \pi x$ is compatible with $p_3(x, 0)$ and is given by

$$p_3(x, 0) = \pi \left[ Q_1^3(x) + Q_1^3(1 - x) \right] - \pi^2 \left[ Q_3^3(x) + Q_3^3(1 - x) \right],$$

the $3^{rd}$ order Hermite interpolant of $\sin \pi x$. We have observed order of errors $O(10^{-3})$ and $O(10^{-5})$ in $p_3(x, t)$ at $t = 1, 5$, with $\alpha = 2$, $j = 1, 2$, $\epsilon = 0.1$. Further, we have verified the Hermite interpolants $p_3(x, t)$ with the finite difference numerical solutions for $\alpha = 2$, $j = 1, 2$ and $\epsilon = 0.05$. In this case also an excellent agreement is observed between the Hermite interpolants $p_3(x, t)$ and the corresponding numerical solution.

We may point out that the Hermite interpolation approximation of the solution of IBVP (1)–(3) can be used for non zero $a, b$ also. The only requirement is the compatibility of initial and boundary data with the given PDE (see (7) and (8)). For $\alpha = 2$,

$$u_{x, A}(0, 0) \neq u_0^{(4)}(0), \quad u_{x, A}(1, 0) \neq u_0^{(4)}(1)$$

when $u_0(x) = \sin(\pi x)$ and $a = b = 0$. That is, the compatibility condition is not satisfied. Therefore, we have not computed $p_4(x, t)$.

### 3 Conclusions

Inspired by Grundy’s (see [3]) idea of using Hermite interpolants to approximate solutions of initial boundary value problems for nonlinear partial differential equations, we have approximated the solution of IBVP (1)–(3) by Hermite interpolants $p_3(x, t)$. In the process we arrived at a system of nonlinear ODEs (9) involving $V_0$ and $V_1$, the unknown fluxes at $x = 0$ and $x = 1$. For $n = 2$, the system of ODEs is explicitly given by (10)–(11) for a specific set of parameter values. The system of ODEs resulting from (9) for different values of $n$ is numerically solved for $V_0$ and $V_1$ subject to initial conditions (8). Then $p_n(x, t)$ is computed at different times and compared with a numerical solution obtained by a finite difference scheme due to Dawson [1]. We have used Hermite interpolants $p_n(x, t)$ of order up to $n = 4$ because of computational simplicity and also because of the fact that $p_4(x, t)$ has agreed with the numerical solution reasonably well.

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### References


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