

Global Existence of Solutions to the Modified Camassa-Holm Shallow Water Equation

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Abstract: This paper studies the global existence of the modified Camassa-Holm equation. We firstly find a conservation law. Then a new global existence result is obtained.

Keywords: Modified Camassa-Holm equation; Conservation law; Global existence

1 Introduction

In this paper, we consider the modified Camassa-Holm shallow water equation [3] derived as the Euler-poincare differential equation on the Bott-virasoro group with respect to the H^k metric. That is the equation

$$m_t + um_x + 2u_x m = 0, \text{ with } m = \Lambda^{2k} u = (1 - \partial_x^2)^k u. \quad (1)$$

where k is a positive integer, $u(t, x)$ describes the horizontal velocity of the fluid.

For $k = 0$, Eq. (1) becomes the Korteweg-de Vries (KdV) equation (see [19, 21]) which describes the unidirectional propagation of waves at free surface of shallow water under the influence of the gravity. The solitary waves of the KdV equation are smooth solitons. For $k = 1$ it is the well-known Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

which is modeling the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction [20-21]. The Camassa-Holm equation has a bi-Hamiltonian structure [10, 14] and is completely integrable [15-16]. Moreover, it has many conservation laws (see [7]):

$$E_1 = \int_R u dx, E_2 = \int_R (u^2 + u_x^2) dx, E_3 = \int_R (u^3 + uu_x^2) dx.$$

Moreover, it has an infinite number of solitary wave solutions [8, 22]. $u(t, x) = ce^{-|x-ct|}$, $c \in R$ called peakons due to the discontinuity of their first derivatives at the wave peak.

The Cauchy problem for the Camassa-Holm equation has been studied extensively [2, 4-6, 9, 12-13, 17-18, 23-24]. It has been shown that this equation is local well-posed for initial data $u_0 \in H^s(R)$, $s > \frac{3}{2}$. Furthermore, the global existence and blow-up for this equation also have been well studied [5-6, 11-12, 17-18, 24]. Here blow-up means that the slope of the solution becomes unbounded while the solution itself stays bounded. Particularly, A. Constantin and J. Escher studied the global and blow-up for Camassa-Holm equation; the well-posedness, global existence and blow-up phenomena for a periodic quasi-linear hyperbolic equation; the blow-up rate and blow-up of breaking waves for the Camassa-Holm equation.

In recent years, many researches have been researched on Camassa-Holm equation. In [3], Robert McLachlan and Xingyou Zhang studied the well-posedness and dynamics of a modified Camassa-Holm equation on the unit circles and got the result that it had some significant differences from that of Camassa-Holm equation. Applying variational derivative approach, R. Naz, I. Naeem and S. Abelman in [7] derived the conservation laws for the Camassa-Holm equation. They only considered the first order multipliers.

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Moreover, two multipliers were obtained for the Camassa-Holm equation. The integrable two-component Camassa-Holm shallow water system was discussed in [12]. G.M. Coclite, H. Holden and K.H. Karlsen considered the well-posedness of higher-order Camassa-Holm equations, which extended the Camassa-Holm equation to the higher-order form. They established the existence of global weak solutions and presented some invariant spaces under the action of the equation. Moreover, a "weak equals strong" uniqueness result was proved (see [4]).

On the bases of the above studies, this paper studies the modified Camassa-Holm equation. The structure of this paper is organized as follows: In section 2, some preliminary knowledges are given. In Section 3, we get a conservation law and a global existence result.

2 Preliminaries

In the following discussion, we consider the case of $k = 2$. Now we will give the main lemmas.

The Cauchy problem of Eq. (1) is

$$\begin{cases} m_t + 2u_x m + u m_x = 0, & t > 0, x \in R, \\ m(0, x) = (1 - \partial_x^2 + \partial_x^4)u_0(x) = \Lambda^4 u_0(x), & x \in R. \end{cases} \quad (2)$$

Eq. (2) also can be rewritten as the following convolution form

$$\begin{cases} u_t = -u u_x - \partial_x \Lambda^{-4}(u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}), & t > 0, x \in R, \\ u(0, x) = u_0(x), & x \in R. \end{cases} \quad (3)$$

Lemma 1 (see [1, 3]) Let $k = 2$, and $u_0 \in L^1(R) \cap H^s(R)$, $s > \frac{7}{2}$. Then there exist a $T > 0$ depending on $\|u_0\|_{H^s(R)}$, and a unique solution $u = u(t, x)$ satisfying Eq. (2) (or Eq. (3)) such that

$$u \in C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R))$$

Moreover, the map $u_0 \in H^s(R) \cap L^1(R) \rightarrow u \in C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R))$ is continuous.

Lemma 2 Given $u_0 \in L^1(R) \cap H^s(R)$, $s > \frac{7}{2}$, and let T be the maximal existence time of the solution $u = u(t, x)$ to Eq. (2) (or Eq. (3)) with the initial u_0 . Then the corresponding solution blows up in finite time if and only if

$$\liminf_{t \rightarrow T} \inf_{x \in R} \{u_x(t, x)\} = -\infty.$$

Proof. Multiplying the first equation in (2) by m and integrating by parts, we have

$$\frac{d}{dt} \int_R m^2(t, x) dx = 2 \int_R m m_t dx = 2 \int_R m(-u m_x - 2u_x m) dx = -3 \int_R m^2 u_x dx.$$

Differentiating the first equation in (2) with respect to x , multiplying the obtained equation by m_x , and integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dt} \int_R m_x^2(t, x) dx &= 2 \int_R m_x m_{tx} dx \\ &= -6 \int_R m_x^2 u_x dx - 2 \int_R u m_x m_{xx} dx - 4 \int_R m m_x u_{xx} dx \\ &= -5 \int_R m_x^2 u_x dx - 4 \int_R m m_x u_{xx} dx \\ &= -5 \int_R m_x^2 u_x dx + 2 \int_R m^2 u_{xxx} dx. \end{aligned}$$

Thus $\frac{d}{dt} \int_R (m^2 + m_x^2) dx = -3 \int_R m^2 u_x dx - 5 \int_R m_x^2 u_x dx + 2 \int_R m^2 u_{xxx} dx$. From [lemma 5.1, 3], we know there exists a constant K , such that $\|u_{xxx}\|_{L^\infty} \leq K$. Here we assume that there exists $M > 0$, such that $u_x(t, x) \geq -M$, $\forall (t, x) \in [0, T) \times R$. From the above, we get

$$\frac{d}{dt} \int_R (m^2 + m_x^2) dx \leq (5M + 2K) \int_R (m^2 + m_x^2) dx.$$

By means of Gronwall's inequality, we have

$$\|u(t, \cdot)\|_{H^3}^2 \leq \|m(t, \cdot)\|_{H^2}^2 \leq \|m(0, \cdot)\|_{H^2}^2 \exp((5M + 2K)t).$$

The above inequality, Sobolev's imbedding theorem and Lemma 2 ensure that the solution u does not blow-up in finite time. So the above assumption is wrong. Hence we get the result of this lemma. This completes the proof of this lemma.

Introduce the following differential equation

$$\begin{cases} q_t = u(t, q), & t \in [0, T], x \in R, \\ q(0, x) = x, & x \in R. \end{cases} \quad (4)$$

Applying classical results in the theory of ordinary differential equation, one can obtain a result on q which is crucial in studying global existence.

Lemma 3 (see [12,13]) *Let $u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$, $s > \frac{7}{2}$. Then Eq. (4) has a unique solution $q \in C^1([0, T] \times R; R)$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of R with*

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x)) ds\right) > 0, \quad \forall (t, x) \in [0, T] \times R.$$

3 Global existence

In this section we derive a conservation law and an a priori estimate for the solution to Eq. (2). Using this conservation law and a priori estimate, we then establish a new global existence result.

Theorem 4 *Let $u_0 \in L^1(R) \cap H^s(R)$, $s > \frac{7}{2}$, and let T be the maximal existence time of the solution $u = u(t, x)$ to Eq. (2) (or Eq. (3)) with the initial u_0 . Then the corresponding solution $u = u(t, x)$ exists globally in time.*

Proof. In order to prove this theorem, we only need the following two lemmas.

Lemma 5 *Let $u_0 \in L^1(R) \cap H^s(R)$, $s > \frac{7}{2}$, and let T be the maximal existence time of the solution $u = u(t, x)$ to Eq. (2) (or Eq. (3)) with the initial u_0 . Then we have*

$$E(t) = \int_R (u^2 + 2u_x^2 + u_{xx}^2) dx = \int_R (u_0^2 + 2u_{0,x}^2 + u_{0,xx}^2) dx, \quad \forall t \in [0, T]. \quad (5)$$

Moreover, we have

$$\|u(t, \cdot)\|_{L^\infty(R)}^2 \leq \|u_0\|_{H^2(R)}^2, \quad \forall t \in [0, T]. \quad (6)$$

proof. Applying Eq. (2) and integrating by parts, we can get

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_R (2uu_t + 4u_xu_{xt} + 2u_{xx}u_{xxt}) dx \\ &= \int_R (2uu_t - 4uu_{xxt} + 2uu_{xxxxt}) dx \\ &= \int_R 2u(u_t - 2u_{xxt} + u_{xxxxt}) dx \\ &= \int_R 2u(1 - \partial_x^2)^2 u_t dx = \int_R 2um_t dx \\ &= \int_R 2u(-um_x - 2u_xm) dx = 0. \end{aligned}$$

This implies $E(t) = E(0)$ for all $t \in [0, T]$. Using this conservation law, we obtain

$$\begin{aligned} u^2(t, \cdot) &\leq \frac{1}{2} \int_R (u^2 + u_x^2) dx \\ &\leq \frac{1}{2} \int_R (u^2 + 2u_x^2 + u_{xx}^2) dx = \frac{1}{2} \int_R (u_0^2 + 2u_{0,x}^2 + u_{0,xx}^2) dx \\ &\leq \frac{1}{2} \cdot 2 \|u_0\|_{H^2}^2 = \|u_0\|_{H^2}^2. \end{aligned}$$

hence, we get

$$\|u(t, \cdot)\|_{L^\infty(R)}^2 \leq \|u_0\|_{H^2(R)}^2, \quad \forall t \in [0, T].$$

This completes the proof of this lemma.

Lemma 6 Let $u_0 \in L^1(R) \cap H^s(R)$, $s > \frac{7}{2}$, and let T be the maximal existence time of the solution $u = u(t, x)$ to Eq. (2) (or Eq. (3)) with the initial u_0 . Then we have

$$\liminf_{t \rightarrow T} \inf_{x \in R} \{u_x(t, x)\} \geq -\frac{\sqrt{2}}{2} (1 + \|u_0\|_{H^2}^2) \exp(18c_0 \|u_0\|_{H^2}^2 T).$$

Proof. By Lemma 3, we know that $q(t, \cdot)$ is an increasing diffeomorphism of R with

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x))\right) > 0, \quad \forall (t, x) \in [0, T] \times R.$$

Then we have

$$\inf_{x \in R} u_x(t, q(t, x)) = \inf_{x \in R} u_x(t, x), \quad \forall t \in [0, T]. \quad (7)$$

Set $M(t, x) = u_x(t, q(t, x))$. By Eq. (4), we have

$$\frac{\partial M}{\partial t} = (u_{tx} + uu_{xx})(t, q(t, x)).$$

Differentiating the first equation in (3) with respect to x , we have

$$u_{tx} + u_x^2 + uu_{xx} + \partial_x^2 \Lambda^{-4} (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}) = 0.$$

Then

$$\begin{aligned} M_t &= -u_x^2 - \partial_x^2 \Lambda^{-4} (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}) \\ &= -u_x^2 - \partial_x^2 (1 - \partial_x^2)^{-2} (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}) \\ &= -u_x^2 - \partial_x^2 p * p * (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}), \end{aligned}$$

where $p = \frac{1}{2}e^{-|x|}$. From [4], we know $p(x) \geq 0$, $\|p(x)\|_{w^{1,1}(R)}$, $\|p(x)\|_{w^{1,\infty}(R)} \leq c_0$, c_0 is a positive constant. Let $f(t, x) = -\partial_x^2 p * p * (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx})$. Using the relation $\partial_x^2 p * f = p * f - f$, we get

$$f(t, x) = p * (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}) - p * p * (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}).$$

We first consider

$$\begin{aligned} & p * (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}) \\ &= \int_R p(x-y) (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx})(t, y) dy \\ &= \int_R p(x-y) (u^2 + 2u_x^2 - 2u_{xx}^2) dy - \int_R p(x-y) (\frac{1}{2}u_{xx}^2 - 5u_x u_{xxx}) dy \\ &\leq \|p(x-y)\|_{L^\infty} \cdot 2 \|u^2 + 2u_x^2 + u_{xx}^2\|_{L^1} + 5 \int_R p(x-y) d(u_x u_{xxx}) \\ &\leq 2c_0 \cdot 2 \|u_0\|_{H^2}^2 + 5 \|\partial_x p\|_{L^\infty} \|u_x^2 + u_{xx}^2\|_{L^1} \\ &\leq 4c_0 \|u_0\|_{H^2}^2 + \frac{5}{2} \cdot 2c_0 \|u_0\|_{H^2}^2 = 9c_0 \|u_0\|_{H^2}^2. \end{aligned}$$

Then we also can get $p * p * (u^2 + 2u_x^2 - \frac{5}{2}u_{xx}^2 - 5u_x u_{xxx}) \leq 9c_0 \|u_0\|_{H^2}^2$. Then

$$|f(t, x)| \leq 18c_0 \|u_0\|_{H^2}^2, \quad \forall (t, x) \in [0, T] \times R. \quad (8)$$

Thus

$$M_t = -u_x^2 + f(t, x) = -M^2 + f(t, x), \quad (t, x) \in [0, T) \times R. \quad (9)$$

Consider the following function

$$w(t, x) = 1 + M^2(t, x), \quad (t, x) \in [0, T) \times R. \quad (10)$$

By Sobolev imbedding theorem, we have

$$0 < w(0, x) = 1 + M^2(0, x) = 1 + u_x^2(0, q(0, x)) = 1 + u_{0,x}^2(x) \leq 1 + \|u_0\|_{H^2}^2. \quad (11)$$

Differentiating (10) with respect to t and using (8) and (9), we obtain

$$\frac{\partial w}{\partial t} = 2MM_t = 2M(-M^2 + f(t, x)) \leq (1 + M^2)|f(t, x)| \leq w(t, x)18c_0 \|u_0\|_{H^2}^2. \quad (12)$$

By Gronwall's inequality and using (11) and (12), we get

$$w(t, x) \leq w(0, x) \exp(18c_0 \|u_0\|_{H^2}^2 t) \leq (1 + \|u_0\|_{H^2}^2) \exp(18c_0 \|u_0\|_{H^2}^2 T),$$

for all $(t, x) \in [0, T) \times R$. On the other hand

$$w(t, x) \geq \frac{1}{2} + M^2(t, x) \geq \sqrt{2}|M(t, x)|, \quad \forall (t, x) \in [0, T) \times R.$$

Thus $M(t, x) \geq -\frac{1}{\sqrt{2}}w(t, x) \geq -\frac{1}{\sqrt{2}}(1 + \|u_0\|_{H^2}^2) \exp(18c_0 \|u_0\|_{H^2}^2 T)$, for all $(t, x) \in [0, T) \times R$. Then by (7) and the above inequalities, we have

$$\inf_{x \in R} u_x(t, q(t, x)) = \inf_{x \in R} u_x(t, x) \geq -\frac{1}{\sqrt{2}}(1 + \|u_0\|_{H^2}^2) \exp(18c_0 \|u_0\|_{H^2}^2 T).$$

This completes the proof of this lemma.

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