Initial Boundary Value Problems for the General Shallow Water Wave Equation without Dispersion Term

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Abstract: In this paper we study initial boundary value problem of general shallow water wave equation without dispersion term on the half-line and on a finite interval subject to homogeneous Dirichlet boundary conditions. Our approach is based on sharp extension results for functions on the half-line or on a finite interval symmetry preserving properties of the equation under discussion.

Keywords: General shallow water wave equation without dispersion term; initial boundary value problems; local well-posedness; Blow-up; global existence.

1 Introduction

In this paper we present a thorough study on initial boundary value problems of general shallow water wave equation without dispersion term on the half-line and on a finite interval. In [1]-[2], Degasperis and Procesi firstly studied the following family of third order dispersive PDE conservation laws

\[ u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left( c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx} \right)_x \]

where \( \alpha, c_0, c_1, c_2 \) and \( c_3 \) are real constants and indices denote partial derivatives. When \( c_1 = -\frac{\alpha}{2} \), \( c_2 = \frac{\varepsilon(\beta-1)}{2} \), \( c_3 = \varepsilon \) and replacing \( c_0 \) with \( k \), and \( \alpha^2 \) with \( \varepsilon \) in the equation above, we obtain the following equation

\[
\begin{cases}
(u - \varepsilon u_{xx})_t + ku_x + \alpha uu_x + \gamma u_{xxx} = \varepsilon(\beta u_x u_{xx} + uu_{xxx}), & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = u_0(x),
\end{cases}
\]

where \( u(x, t) \) stands for the fluid velocity in the \( x \) direction (or equivalently the height of the free surface of water above a float bottom), \( k \) is a constant related to the critical shallow water wave speed, and \( \alpha, \beta, \varepsilon \) are dispersion parameters. It is necessary to point out that Eq.(2) is equivalent to Eq.(1) since when \( \varepsilon = \alpha^2 = c_3, k = c_0, \alpha = -2c_1 \) and \( \beta = 1 + \frac{2c_2}{\varepsilon} \), Eq.(2) turns out to be Eq.(1). In Eq.(1.2), \( u_x \) and \( u_{xxx} \) are dispersion terms. When \( k = \gamma = 0 \), Eq.(2) change into

\[
\begin{cases}
(u - \varepsilon u_{xx})_t + \alpha uu_x = \varepsilon(\beta u_x u_{xx} + uu_{xxx}), & x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = u_0(x),
\end{cases}
\]

we call Eq(3)the general shallow water wave equation without dispersion term. In Eq.(2), if \( \alpha - \beta = 1, \gamma = -k\varepsilon \), then this equation includes the KdV equation, the Camassa-Holm equation, the Degasperis-Procesi equation and the \( b \)-family of equations.

The remainder of the paper is organized as follows: In section 2, we derive sharp extension results, which are crucial for our approach. In section 3, by using the conservation of symmetr enjoyed by the equation (3), we study initial boundary value problems of the equation (3) on the half-line and on the finite interval. In section 4, we investigate blow-up and the global existence of solutions to equation (3).

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2 Several key lemmas

In the section we present several technical lemmas which will be crucial for our purpose.

**Lemma 1** Given $s \in \left(\frac{1}{2}, \frac{5}{2}\right)$. Assume that $v \in H^s(R_+)$ with $v(0) = 0$. Let furthermore
\[
\tilde{v}(x) = \begin{cases} 
  v(x), & x \geq 0 \\
  -v(-x), & x < 0 
\end{cases}, \text{ then } \tilde{v} \in H^s(R).
\]

**Remark 1** For $s \geq \frac{5}{2}$, under the same assumption of Lemma 1, one can not deduce $\tilde{v} \in H^s(R)$ generally.

In order to obtain $\tilde{v} \in H^s(R)$, one has to add additional conditions. For this we let $k = 0, 1, 2, \ldots$, and for $2k + \frac{1}{2} < s < 2k + \frac{5}{2}$, we set $D^k_s(R_+) = \{v \in H^s(R_+)|v(2k)(0) = v(2k-2)(0) = \ldots = v(0) = 0\}$. We now have the following lemma:

**Lemma 2** Assume that $v \in D^k_s(R_+)$, where $k = 0, 1, 2, \ldots$, and $2k + \frac{1}{2} < s < 2k + \frac{5}{2}$. Let furthermore
\[
\tilde{v}(x) = \begin{cases} 
  v(x), & x \geq 0 \\
  -v(-x), & x < 0 
\end{cases}, \text{ then } \tilde{v} \in H^s(R).
\]

**Lemma 3** Given $s \in [0, 1]$. Assume that $v \in H^s(0, l)$ with $v(0) = 0$. Let furthermore
\[
\tilde{v}(x) = \begin{cases} 
  v(x), & x \in [0, l] \\
  -v(-x), & x \in (-l, 0) 
\end{cases}, \text{ then } \tilde{v} \in H^s(-l, l).
\]

**Remark 2** For $s \geq \frac{5}{2}$, under the same assumption of Lemma 3, one can not deduce $\tilde{v} \in H^s(-l, l)$ generally.

In order to obtain $\tilde{v} \in H^s(-l, l)$, one has to add additional conditions. For this we let $k = 0, 1, 2, \ldots$, and for $2k + \frac{1}{2} < s < 2k + \frac{5}{2}$, we set $D^k_s(0, l) = \{v \in H^s(0, l)|v(2k)(0) = v(2k)(l) = v(2k-2)(0) = \ldots = v(0) = v(l) = 0\}$. We now have the following lemma:

**Lemma 4** Assume that $D^k_s(0, l) = \{v \in D^k_s(0, l)\}$, where $k = 0, 1, 2, \ldots$, and $2k + \frac{1}{2} < s < 2k + \frac{5}{2}$. Let furthermore $\tilde{v}(x) = \begin{cases} 
  v(x), & x \geq 0 \\
  -v(-x), & x < 0 
\end{cases}$, then $\tilde{v}(x)$ belongs to $v \in H^s(-l, l)|v(2k)(-l) = v(2k)(l) = v(2k-2)(-l) = v(2k-2)(l) = \ldots = v(-l) = v(l) = 0$.

The proof of Lemma 1-Lemma 4 is given in [4].

3 The General Shallow Water Wave Equation without dispersion term

The well-posedness result of the equation (2) is true [2]. So the well-posedness result of the equation (3) is obvious. In this section, we will investigate initial boundary value problems of the equation (3) on the half-line and on the finite interval:

3.1 The case of the half-line

Let us consider the following initial boundary value problem equation of (3) on the half-line:
\[
\begin{align*}
(u - \varepsilon u_{xx})_t + \alpha u_{xx} &= \varepsilon(\beta u_x u_{xx} + uu_{xxx}), & t > 0, & x \in R_+, \\
u(0, x) &= u_0(x), & x \in R_+, \\
u(t, 0) &= 0, & t \geq 0.
\end{align*}
\]

**Theorem 5** Assume that $u_0 \in H^s(R_+) \cap H^1_0(R_+)$, with $\frac{3}{2} < s < \frac{5}{2}$. Then there exists a maximal $T = T(u_0) > 0$ and a unique solution $u(t, x)$ to Eq.(4) such that $u = u(\cdot, u_0)$ belongs to
\[
C \left(\left[0, T\right]; H^s(R_+) \cap H^1_0(R_+)\right) \cap C^1 \left(\left[0, T\right]; H^{s-1}(R_+) \cap H^1_0(R_+)\right).
\]

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(R_+) \cap H^1_0(R_+) \mapsto C \left(\left[0, T\right]; H^s(R_+) \cap H^1_0(R_+)\right) \cap C^1 \left(\left[0, T\right]; H^{s-1}(R_+) \cap H^1_0(R_+)\right)$ is continuous.
Following a similar argument as in Theorem 5, we first extend the initial data \( u_0(x) \) defined on the half-line into an odd function defined on the line:

\[
\tilde{u}_0(x) = \begin{cases} 
  u_0(x), x \geq 0 \\
  -u_0(-x), x < 0 
\end{cases}
\] (6)

Note that \( u_0(x) \in H^s(R_+) \cap H_0^1(R_+) \) with \( \frac{3}{2} < s < \frac{5}{2} \). The relation (6) and Lemma 1 show that \( \tilde{u}_0(x) \in H^s(R) \) with \( \frac{3}{2} < s < \frac{5}{2} \) is an odd function.

We now can convert Eq.(4) into the General Shallow Water Wave Equation without dispersion term on the whole line.

\[
\begin{align*}
& (\tilde{u} - \varepsilon \tilde{u}_{xx})_t + \alpha \tilde{u}_x = \varepsilon (\beta \tilde{u}_x \tilde{u}_{xx} + \tilde{u}\tilde{u}_{xxx}), t > 0, x \in R, \\
& \tilde{u}(0, x) = \tilde{u}_0(x), \text{ (odd)}, x \in R.
\end{align*}
\] (7)

Apply the previous local well-posedness result of the Cauchy problem for the equation (2) on the line [5], we conclude that there exists a maximal \( T = T(\tilde{u}_0) > 0 \) and a unique solution \( \tilde{u}(t,x) \) to Eq.(7) such that \( \tilde{u} = \tilde{u}(. , \tilde{u}_0) \in C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R)) \). Moreover, the solution depends continuously on the initial data, i.e. the mapping \( \tilde{u}_0 \mapsto \tilde{u}( ., \tilde{u}_0) : H^s(R) \to C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R)) \) is continuous.

In addition, if \( \tilde{u}(t,x) \) is a solution to Eq.(7), then it is not difficult to check that the function \( \tilde{u}_1(t,x) := -\tilde{u}(t,-x), (t,x) \in [0,T] \times \mathbb{R} \) is also a solution of Eq.(7) in \( C([0, T]; H^s(R)) \cap C^1([0, T]; H^{s-1}(R)) \) with the same initial data \( \tilde{u}_0 \). By uniqueness we conclude that \( \tilde{u}_1 \equiv \tilde{u} \). Thus \( \tilde{u}(t,x) \) is odd for any \( t \in [0,T) \). In particular, we have \( \tilde{u}(t,0) \equiv 0 \) for all \( t \in [0,T) \). Set \( u(t,x) = \tilde{u}(t,x) \) for all \( t \in [0,T] \times R_+ \). Then we see that \( u(t,x) \in C([0, T]; H^s(R_+) \cap H_0^1(R_+)) \cap C^1([0, T]; H^{s-1}(R_+) \cap H_0^1(R_+)) \) is a solution to Eq.(4). On the other hand, if \( v(t,x) \) is also a solution to Eq.(4) with the same initial data \( u_0(x) \), then

\[
\tilde{v}(x) = \begin{cases} 
  v(t,x), x \geq 0 \\
  -v(t,-x), x < 0 
\end{cases}
\] (8)

is the unique solution in (5) to Eq.(7) with the initial data \( u_0(x) \). By the uniqueness, we conclude that \( u(t,x) = v(t,x) \) is the unique solution to Eq.(4) with the initial data \( u_0(x) \). Obviously, the continuity of \( \tilde{u}_0 \to \tilde{u}(., \tilde{u}_0) \) implies that of \( u_0 \to u(., u_0) \) as well. \( \blacksquare \)

**Remark 3**  Assume that \( u_0(x) \in H^s(R_+) \cap H_0^1(R_+) \) with \( s \geq \frac{5}{2} \). Thus, given \( r \in (\frac{3}{2}, \frac{5}{2}) \), it follows from Theorem 5 that there exists a maximal \( T = T(u_0) > 0 \) and a unique solution \( u(t, x) \) to Eq. (4) in the class \( C([0, T]; H^r(R_+) \cap H_0^1(R_+)) \cap C^1([0, T]; H^{r-1}(R_+) \cap H_0^1(R_+)) \). However, the arguments of the proof of Theorem 5 cannot be used to conclude that \( u \in C([0, T]; H^s(R_+) \cap H_0^1(R_+)) \cap C^1([0, T]; H^{s-1}(R_+) \cap H_0^1(R_+)) \).

In order to study more regular solutions, we may consider the following initial boundary value problem:

\[
\begin{align*}
& (u - \varepsilon u_{xx})_t + \alpha uu_x = \varepsilon (\beta u_x uu_{xx} + uu_{xxx}), t > 0, x \in R_+, \\
& u(0, x) = u_0(x), \text{ (odd)}, x \in R_+, \\
& u(2k)(t, 0) = u(2k-2)(t, 0) = ... = u(t, 0) = 0, t \geq 0
\end{align*}
\] (9)

We next present the following local well-posedness result.

**Theorem 6**  Assume that \( u_0 \in D^k_k(R_+), \text{ where } k = 1, 2, ..., \text{ and } 2k + \frac{1}{2} < s < 2k + \frac{5}{2} \). Then there exists a maximal \( T = T(u_0) > 0 \), and a unique solution \( u(t, x) \) to Eq. (9) such that \( u = u(., u_0) \in C([0, T]; D^k_k(R_+)) \cap C^1([0, T]; D^{k-1}_k(R_+)) \). Moreover, the solution depends continuously on the initial data, i.e. the mapping \( u_0 \mapsto u(., u_0) : D^k_k(R) \to C([0, T]; D^k_k(R_+) \cap C^1([0, T]; D^{k-1}_k(R_+)) \) is continuous.

**Proof.**  Following a similar argument as in Theorem 5, we first extend the initial data \( u_0(x) \) defined on the half-line into an odd function \( \tilde{u}_0(x) \) defined in (6) on the line. Since \( u_0 \in D^k_k(R_+), \text{ Lemma 2 implies that } \tilde{u}_0(x) \in H^s(R) \) is an odd function. The conclusions follow now as in Theorem 5. \( \blacksquare \)

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3.2 The case of the interval \((0, \frac{1}{2})\)

In this subsection, we study initial boundary value problems of General Shallow Water Wave Equation without dispersion term on the interval \((0, \frac{1}{2})\). More precisely, let us consider the following problem:

\[
\begin{cases}
(u - \varepsilon u_{xx})_t + \alpha uu_x = \varepsilon(\beta u_x u_{xx} + uu_{xxx}), t > 0, x \in (0, \frac{1}{2}), \\
u(0, x) = u_0(x), \quad (\text{odd}), \quad x \in (0, \frac{1}{2}), \\
u(t, 0) = u(t, \frac{1}{2}) = 0, \quad t \geq 0.
\end{cases}
\]

(10)

We first present the local well-posedness result for Eq. (10).

**Theorem 7** Assume that \(u_0(x) \in H^s((0, \frac{1}{2})) \cap H^1_0((0, \frac{1}{2})) \) with \(\frac{3}{2} < s < \frac{5}{2}\). Then there exists a maximal \(T = T(u_0) > 0\) and a unique solution \(u(t, x)\) to Eq.(10) such that \(u = u(., u_0)\) belongs to

\[
C \left([0, T) ; H^s((0, \frac{1}{2}) \cap H^1_0((0, \frac{1}{2})) \right) \cap C^1 \left([0, T) ; H^{s-1}((0, \frac{1}{2}) \cap H^1_0((0, \frac{1}{2})) \right)
\]

(11)

Moreover, the solution depends continuously on the initial data, i.e. the mapping \(u_0 \mapsto u(., u_0) : H^s((0, \frac{1}{2}) \cap H^1_0((0, \frac{1}{2})) \to C \left([0, T) ; H^s((0, \frac{1}{2}) \cap H^1_0((0, \frac{1}{2})) \cap C^1 \left([0, T) ; H^{s-1}((0, \frac{1}{2}) \cap H^1_0((0, \frac{1}{2})) \right) \text{is continuous.}
\]

In order to proof this Theorem, first we need to prove the local well-posedness result of the periodic General Shallow Water Wave Equation.

**Lemma 8** Given \(u_0 \in H^s(S), \ s > \frac{3}{2}, \) with \(S = R/Z\) (the circle of unite length), then there exists a maximal \(T = T(u_0) > 0\) and a unique solution \(u(t, x)\) to

\[
\begin{cases}
(u - \varepsilon u_{xx})_t + ku_x + \alpha uu_x + \gamma u_{xxx} = \varepsilon(\beta u_x u_{xx} + uu_{xxx}), x \in R, t > 0, \\
u(0, x) = u_0(x), x \in R, \\
u(t, x) = u(t, x + 1), t \geq 0, x \in R,
\end{cases}
\]

(12)

such that \(u = u(., u_0)\) belongs to \(C \left([0, T) ; H^s(S) \right) \cap C^1 \left([0, T) ; H^{s-1}(S) \right). \) Moreover, the solution depends continuously on the initial data, i.e. the mapping \(u_0 \mapsto u(., u_0) : H^s(S) \to C \left([0, T) ; H^s(S) \right) \cap C^1 \left([0, T) ; H^{s-1}(S) \right) \text{is continuous.}
\]

The proof of Lemma 8 is similar to the well-posedness result of the General Shallow Water Wave Equation on the line[5], so we omit it.

**Proof of Theorem 7:** We first convert the initial boundary value problem of the equation (3) on the interval \((0, \frac{1}{2})\) into the Cauchy problem of the periodic equation (3) with period 1. In order to do so, we extend the initial data \(u_0(x)\) defined on the interval \((0, \frac{1}{2})\) into a periodic odd function defined on the line:

\[
\tilde{u}_0(x) = \begin{cases}
u_0(x), x \in [n, \frac{1}{2} + n], n \in Z, \\-\nu_0(-x), x \in [-\frac{1}{2}, n], n \in Z.
\end{cases}
\]

(13)

Note that \(u_0(x) \in H^s((0, \frac{1}{2}) \cap H^1_0((0, \frac{1}{2}) \) with \(\frac{3}{2} < s < \frac{5}{2}\). Then we have

\[
\lim_{x \to \frac{1}{2}^-} \frac{\tilde{u}_0(x)}{x - \frac{1}{2}} = \lim_{x \to \frac{1}{2}^+} \frac{\tilde{u}_0(-x)}{x - \frac{1}{2}} = \lim_{x \to \frac{1}{2}^-} \frac{\tilde{u}_0(\frac{1}{2})}{x - \frac{1}{2}} = \lim_{x \to \frac{1}{2}^+} \frac{\tilde{u}_0(\frac{1}{2})}{x - \frac{1}{2}}
\]

Combining the above relation with (13) and Lemma 3 with \(l = \frac{1}{2}\), we have that \(\tilde{u}_0(x) \in H^s(-\frac{1}{2}, \frac{1}{2}) \cap H^1_0((-\frac{1}{2}, \frac{1}{2}) \) with \(\frac{3}{2} < s < \frac{5}{2}\) is a periodic odd function.

Thus, we may convert the equation (3) on the interval \((0, \frac{1}{2})\) into the following periodic problem:

\[
\begin{cases}
(\tilde{u} - \varepsilon \tilde{u}_{xx})_t + \alpha \tilde{u}_x = \varepsilon(\beta \tilde{u}_x \tilde{u}_{xx} + \tilde{u}_{xxx}), t > 0, x \in R, \\
\tilde{u}_0(0, x) = \tilde{u}_0(x), \quad (\text{odd}), \quad x \in R, \\
\tilde{u}_0(0) = \tilde{u}_0(\frac{1}{2}) = 0, \\
\tilde{u}(t, x) = \tilde{u}_0(t, x + 1) = 0, t \geq 0, x \in R.
\end{cases}
\]

(14)
Applying the local well-posedness result of the periodic General Shallow Water Wave Equation, (Lemma 8), we have that there exists a maximal $T(\tilde{u}_0) > 0$ and a unique solution $\tilde{u}(t, x)$ to Eq.(14) such that $\tilde{u} = \tilde{u}(\cdot, \tilde{u}_0)$ belongs to

$$C \left( \left[ 0, T \right); H^s(0, 1) \cap H^1_0(0, 1) \right) \cap C^1 \left( \left[ 0, T \right); H^{s-1}(0, 1) \cap H^1_0(0, 1) \right)$$  \hspace{1cm} (15)

Moreover, the solution depends continuously on the initial data, i.e. the mapping $\tilde{u}_0 \mapsto \tilde{u}(\cdot, \tilde{u}_0) : H^s(0, 1) \cap H^1_0(0, 1) \to C \left( \left[ 0, T \right); H^s(0, 1) \cap H^1_0(0, 1) \right) \cap C^1 \left( \left[ 0, T \right); H^{s-1}(0, 1) \cap H^1_0(0, 1) \right)$ is continuous.

Note that if $\tilde{u}(t, x)$ is a solution to Eq.(14), then one can check that the function $v(t, x) := -\tilde{u}(t, -x)$, $(t, x) \in \left( 0, T \right) \times \mathbb{R}$ is also a solution of Eq. (14) satisfying (15) with the initial data $\tilde{u}_0$. By uniqueness we conclude that $v \equiv \tilde{u}$ and $\tilde{u}(t, \cdot)$ is odd for any $t \in \left[ 0, T \right)$. Therefore, we have $\tilde{u}(t, 0) \equiv 0$ for all $t \in \left[ 0, T \right)$.

Set $u(t, x) = \tilde{u}(t, x)$ for all $t \in \left[ 0, T \right)$ and $x \in \left[ 0, \frac{1}{2} \right)$. Then we know that $u(x, t) \in C \left( \left[ 0, T \right); H^s(0, 1) \cap H^1_0(0, 1) \right) \cap C^1 \left( \left[ 0, T \right); H^{s-1}(0, 1) \cap H^1_0(0, 1) \right)$ is a solution to Eq. (10). On the other hand, if $u$ is a solution to Eq. (10) with the initial data $u_0(x)$, then

$$\tilde{u}(x) = \left\{ \begin{array}{l} v(t, x), x \in \left[ n, \frac{1}{2} + n \right], n \in \mathbb{Z}, \\ -v(t, -x), x \in \left[ n - \frac{1}{2}, \frac{1}{2} - n \right], n \in \mathbb{Z}. \end{array} \right. \hspace{1cm} (16)$$

is the unique solution to E.q. (14) with the initial data $\tilde{u}_0$ satisfying (15). By the uniqueness of $\tilde{u}(t, x)$, we conclude that $u(t, x) = v(t, x)$ is the unique solution to E.q. (10) with the initial data $u_0(x)$. This completes the proof of the theorem.

As in the case on the line, our method of proof is not suitable to study more regular solution in the class $C \left( \left[ 0, T \right); H^s(0, 1) \cap H^1_0(0, 1) \right) \cap C^1 \left( \left[ 0, T \right); H^{s-1}(0, 1) \cap H^1_0(0, 1) \right)$ with $s > \frac{3}{2}$.

But, we may consider the following initial boundary value problem:

$$\left\{ \begin{array}{l} (u - \varepsilon u_{xx})_t + \alpha uu_x = \varepsilon (\beta u_x u_{xx} + uu_{xxx}), t > 0, x \in \left[ 0, \frac{1}{2} \right], \\ u(0, x) = u_0(x), (odd), x \in \left[ 0, \frac{1}{2} \right], \\ u^{(2k)}(t, 0) = u^{(2k-2)}(t, 0) = \ldots = u(t, 0) = 0, t \geq 0, \\ u^{(2k)}(t, \frac{1}{2}) = u^{(2k-2)}(t, \frac{1}{2}) = \ldots = u(t, \frac{1}{2}) = 0, t \geq 0, \end{array} \right. \hspace{1cm} (17)$$

**Theorem 9** Assume that $u_0 \in D^k_0(0, \frac{1}{2})$, where $k = 1, 2, \ldots, \text{and } 2k + \frac{1}{2} < s < 2k + \frac{3}{2}$ Then there exists a maximal $T(T(u_0) > 0$ and a unique solution $u(t, x)$ to Eq. (17) such that $u = u(\cdot, u_0) \in C \left( \left[ 0, T \right); D^k_0(0, \frac{1}{2}) \right) \cap C^1 \left( \left[ 0, T \right); D^{k-1}_0(0, \frac{1}{2}) \right)$. Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : D^k_0(0, \frac{1}{2}) \to C \left( \left[ 0, T \right); D^k_0(0, \frac{1}{2}) \right) \cap C^1 \left( \left[ 0, T \right); D^{k-1}_0(0, \frac{1}{2}) \right)$ is continuous.

**Proof.** Following a similar argument as in Theorem 7, we first extend the initial data $u_0(x)$ defined on the interval $\left( 0, \frac{1}{2} \right)$ into an odd function $\tilde{u}_0(x)$ defined in (13) on the line. Since $u_0 \in D^k_0(0, \frac{1}{2})$, Lemma 4 and (13) shows that $\tilde{u}_0(x) \in D^s(0, \frac{1}{2})$ is a periodic odd function. Thus the result follows from Theorem 7.

## 4 Blow up problem and Global existence

In this section, we present blow-up scenario of strong solutions to E.q. (4) and E.q. (10)

**Theorem 10** Given $u_0(x) \in H^s(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+), \frac{3}{2} < s < \frac{5}{2}$, blow up of the solution $u = (\cdot, u_0)$ to E.q.(4) in finite time $T < +\infty$ occurs if and only if $\lim_{t \uparrow T} \inf_{x \in \mathbb{R}_+} \inf_{x \in \mathbb{R}_+} \left\{ u_x(t, x) \right\} = -\infty$.

**Proof.** As before, we first extend the initial data $u_0(x)$ defined on the half-line into an odd function $\tilde{u}_0(x)$ defined in (6) on the line. By Theorem 5, we can obtain the odd solution $\tilde{u}(t, x)$ which is the corresponding strong solution to E.q(7) with the initial data $\tilde{u}_0(x)$. Moreover, $u(t, x) = \tilde{u}(t, x)$ confined on $[0, T) \times \mathbb{R}_+$ is the unique solution to E.q(4) with the initial data $u_0(x)$.

For the General Shallow Water Wave Equation without dispersion term on the line [5], we know that blow up of the solution $\tilde{u} = \tilde{u}(\cdot, u_0)$ to E.q(4) in finite time $T < +\infty$ occurs if and only if $\lim_{t \uparrow T} \inf_{x \in \mathbb{R}} \left\{ u_x(t, x) \right\} = -\infty$. Since $\tilde{u}(t, \cdot)$ is odd, it follows that $\tilde{u}_x(t, \cdot)$ is even. Thus, we have that $\tilde{u}_x(t, \cdot)$
lim inf \{ \inf_{x \in \mathbb{R}} [\tilde{u}_x(t,x)] \} = \lim inf \{ \inf_{x \in \mathbb{R}} [u_x(t,x)] \}. The above two relations imply the desired result of theorem. 

In order to get the blow up result to E.q. (10), we first proof the blow up result of the periodic General Shallow Water Wave Equation.

**Lemma 11** Given \( u_0(x) \in H^s(S), s > \frac{3}{2} \), blow up of the solution \( u = (\cdot, u_0) \) to E.q(2) in finite time \( T < +\infty \) occurs if and only if \( \lim sup_{t \uparrow T} \{ \inf_{x \in \mathbb{R}} [u_x(t,x)] \} = -\infty. \)

The proof of Lemma 11 is similar to that of Theorem 3.6. in [5], so we omit it.

**Theorem 12** Given \( u_0(x) \in H^s(0, \frac{1}{2}) \cap H^s_1(0, \frac{1}{2}) \) with \( \frac{3}{2} < s < \frac{5}{2} \), blow up of the solution \( u = (\cdot, u_0) \) to Eq.(10) in finite time \( T < +\infty \) occurs if and only if \( \lim sup_{t \uparrow T} \{ \inf_{x \in [0, \frac{1}{2}]} [u_x(t,x)] \} = -\infty \).

**Proof.** As befor, we first extend the initial data \( u_0(x) \) defined on the interval \( [0, \frac{1}{2}] \) into an periodic odd function \( \tilde{u}_0(x) \) defined in (13) on the line. By Theorem 7, we can obtain the odd solution \( \tilde{u}(t,x) \) which is the corresponding strong solution to E.q(12) with the initial data \( \tilde{u}_0(x) \). Moreover, \( u(t,x) = \tilde{u}(t,x) \) confined on \( [0,T_\ast] \times [0, \frac{1}{2}] \) is the unique solution to E.q(10) with the initial data \( u_0(x) \).

For the blow up result to periodic General Shallow Water Wave Equation without dispersion term (Lemma 11) we know that blow up of the solution \( \tilde{u} = \tilde{u}(\cdot, \tilde{u}_0) \) to E.q(14) in finite time \( T < +\infty \) occurs if and only if \( \lim sup_{t \uparrow T_\ast} \{ \inf_{x \in [0, \frac{1}{2}]} [\tilde{u}_x(t,x)] \} = -\infty. \) Since \( \tilde{u}(t,\cdot) \) is odd, it follows that \( \tilde{u}_x(t,\cdot) \) is even. Thus, we have that \( \lim sup_{t \uparrow T_\ast} \{ \inf_{x \in [0, \frac{1}{2}]} [\tilde{u}_x(t,x)] \} = \lim sup_{t \uparrow T} \{ \inf_{x \in [0, \frac{1}{2}]} [u_x(t,x)] \}. \) The above two relations imply the desired result of theorem. 

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**References**


