

A Polynomial Expansion Method and Its Application to Nonlinear Equal Width Wave Equation

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Abstract: In this paper, a polynomial expansion method with a computerized symbolic computation is used for constructing new periodic wave solutions for nonlinear Equal width wave equation arising in mathematical physics. As a result, many exact traveling wave solutions are successfully obtained. The method can also be applied to other nonlinear evolution equations.

Keywords: Equal width wave equation; exact solution; polynomial expansion method; homogeneous balance method

1 Introduction

Looking for exact solutions to nonlinear partial differential equations (NLPDEs) has long been a major concern for both mathematicians and physicists. These solutions may well describe various phenomena in many fields, such as hydrodynamic, plasma physics, nonlinear optic, chemistry, biology, etc. Many powerful methods for obtaining exact solutions of NLPDEs have been presented, such as extended mapping method [1], tanh-function method [2], improved projective Riccati method [3], Exp-function method [4], algebraic method [5], Hirota method [6], F-expansion method [7], and so on.

In the present paper, we consider the exact solutions of Equal width wave equation. Equal width wave equation represents the motion of nonlinear dispersive waves which are often encountered in a number of important physical phenomena such as shallow water, ion acoustic plasmas. In 2005, Abdulkadir Dogan [8] solved Equal width wave equation using the Galerkin approach with linear finite elements, and obtain the numerical solution. In the paper [9], Zhao and Rui obtain the periodic solutions of Jacobin elliptic-function type, many kinds of soliton solutions and triangular function solutions. Now following in the present paper, we use a polynomial expansion method with a computerized symbolic computation for solving new periodic wave solutions for nonlinear Equal width wave equation arising in mathematical physics, and obtain many exact traveling wave solutions.

2 A simple description of polynomial expansion method

For a given nonlinear evolution equations with independent variables x and t

$$H(u, u_x, u_t, u_{xx}, \dots) = 0, \quad (1)$$

make the traveling wave transformation

$$u(x, t) = u(\xi), \xi = k(x - \lambda t), \quad (2)$$

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where k and λ are constants to be determined later, and assume $k > 0$. Substituting (2) into (1) yields a ordinary differential equation:

$$h(u, u', u'', \dots) = 0 \quad (3)$$

where $'$ denotes $\frac{d}{d\xi}$. Seek the traveling wave solutions of Eq. (3) expressed in the form

$$u(x, t) = u(\xi) = a_0 + \sum_{j=1}^n [a_j Y^j(\xi) + b_j Y^{-j}(\xi)] \quad (4)$$

where Y satisfy a Riccati equation

$$\frac{dY}{d\xi} = A + BY + CY^2 \quad (5)$$

A, B, C are all parameters, $a_0, a_j, b_j (j = 1, 2, \dots, n)$ are constants to be determined later. The Riccati equation (5) has the following general solutions:

Case I $C = 0, A \neq 0, B \neq 0$:

$$Y = \exp(B\xi) - \frac{A}{B}. \quad (6)$$

Case II $C = 1, A \neq 0, B = 0$:

$$Y = \begin{cases} \sqrt{A} \tan(\sqrt{A}\xi) \\ -\sqrt{A} \cot(\sqrt{A}\xi) \end{cases}, A > 0 \quad \text{or} \quad Y = \begin{cases} -\sqrt{-A} \tanh(\sqrt{-A}\xi) \\ -\sqrt{-A} \coth(\sqrt{-A}\xi) \end{cases}, A < 0. \quad (7)$$

Case III $C = 1, A = 0, B \neq 0$:

$$Y = \frac{B}{c_0 \exp(-B\xi) - 1}, \quad (8)$$

where c_0 is an integration constant.

Case IV $C = 0, A \neq 0, B \neq 0$:

$$Y = \frac{Y_1 - Y_2 C_1 \exp[(Y_1 - Y_2)\xi]}{1 - C_1 \exp[(Y_1 - Y_2)\xi]}, \quad (9)$$

where c_1 is an integration constant, Y_1, Y_2 satisfy: $A + BY + Y^2 = 0$. That is to say, $Y_{1,2} = \frac{-1 \pm \sqrt{B^2 - 4A}}{2}$, $B^2 - 4A > 0$.

Step (1): Determine the integer n by balancing the highest order nonlinear term and the highest order derivative in Eq. (3).

Step (2): Substitute Eq. (4) with the aid of Eq. (5) into Eq. (3) and collect coefficients $Y^j(\xi) (j = 0, 1, \dots, n)$, then set each coefficients to zero to derive a set of over-determined algebraic equations for $a_0, A, B, C, a_j, b_j (j = 1, \dots, n)$.

Step (3): Solve the system of over-determined algebraic equations obtained in Step 2 using the mathematica, we can obtain many exact solutions of Eq. (1) according to Eq.(4), (6), (7), (8) and (9).

3 Exact traveling solutions of Equal width wave equation

Considering the Equal width wave equation

$$u_t + uu_x + \alpha u_{xxt} = 0 \quad (10)$$

Substituting (2) into Eq. (10), integrating it once and setting the integrating constant to be equal to zero, we have

$$-\lambda u + \frac{1}{2}u^2 - \alpha \lambda k^2 u'' = 0. \quad (11)$$

By balancing the highest order nonlinear term u^2 and the highest order derivative term u'' in Eq. (11), we obtain $n = 2$. So we suppose that the solution of Eq. (11) can be expressed by

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}. \quad (12)$$

Substitute (12) with the aid of Eq. (5) into Eq. (11) and collect coefficients $Y^j(\xi)$ ($j = 0, 1, \dots, n$), then set each coefficients to zero to derive a set of over-determined algebraic equations for a_0, A, B, C, a_j, b_j ($j = 1, \dots, n$). Solving the system of over-determined algebraic equations using mathematica, we can obtain many exact solutions of Equal width wave equation.

Case I $C = 0, A \neq 0, B \neq 0$:

$$a_0 = 2\lambda, a_1 = a_2 = 0, b_1 = \frac{12\lambda}{AB}, b_2 = \frac{12\lambda}{B^2}, k = \sqrt{\frac{1}{B^2\alpha}}, (\alpha > 0)$$

$$a_0 = 0, a_1 = a_2 = 0, b_1 = -\frac{12\lambda}{AB}, b_2 = -\frac{12\lambda}{B^2}, k = \sqrt{-\frac{1}{B^2\alpha}}, (\alpha < 0)$$

where λ is left as a free parameter.

Substituting these values into Eq. (12) and Eq. (6), we obtain exact traveling solution of Eq. (1):

$$u_1(x,t) = 2\lambda + \frac{12\lambda}{AB} \left[\exp \frac{B}{\sqrt{B^2\alpha}}(x - \lambda t) - \frac{A}{B} \right]^{-1} + \frac{12\lambda}{B^2} \left[\exp \frac{B}{\sqrt{B^2\alpha}}(x - \lambda t) - \frac{A}{B} \right]^{-2}, (\alpha > 0)$$

$$u_2(x,t) = -\frac{12\lambda}{AB} \left[\exp \frac{B}{\sqrt{-B^2\alpha}}(x - \lambda t) - \frac{A}{B} \right]^{-1} - \frac{12\lambda}{B^2} \left[\exp \frac{B}{\sqrt{-B^2\alpha}}(x - \lambda t) - \frac{A}{B} \right]^{-2}, (\alpha < 0)$$

Case II $C = 1, A \neq 0, B = 0$:

$$a_0 = 3\lambda, a_1 = 0, a_2 = \frac{3\lambda}{A}, b_2 = b_1 = 0, k = \sqrt{\frac{1}{4A\alpha}}, (A > 0, \alpha > 0 \text{ or } A < 0, \alpha < 0)$$

$$a_0 = -\lambda, a_1 = 0, a_2 = -\frac{3\lambda}{A}, b_2 = b_1 = 0, k = \sqrt{-\frac{1}{4A\alpha}}, (A > 0, \alpha < 0 \text{ or } A < 0, \alpha > 0)$$

$$a_0 = 3\lambda, a_1 = a_2 = 0, b_1 = 0, b_2 = \frac{3\lambda}{A}, k = \sqrt{\frac{1}{4A\alpha}}, (A > 0, \alpha > 0 \text{ or } A < 0, \alpha < 0)$$

$$a_0 = -\lambda, a_1 = a_2 = 0, b_1 = 0, b_2 = -\frac{3\lambda}{A}, k = \sqrt{-\frac{1}{4A\alpha}}, (A > 0, \alpha < 0 \text{ or } A < 0, \alpha > 0)$$

$$a_0 = \frac{3}{2}\lambda, a_1 = 0, a_2 = \frac{3\lambda}{4A}, b_1 = 0, b_2 = \frac{3\lambda}{4A}, k = \sqrt{\frac{1}{16A\alpha}}, (A > 0, \alpha > 0 \text{ or } A < 0, \alpha < 0)$$

$$a_0 = \frac{1}{2}\lambda, a_1 = 0, a_2 = -\frac{3\lambda}{4A}, b_1 = 0, b_2 = -\frac{3\lambda}{4A}, k = \sqrt{-\frac{1}{16A\alpha}}, (A > 0, \alpha < 0 \text{ or } A < 0, \alpha > 0)$$

where λ is left as a free parameter.

Substituting these values into Eq. (12) and Eq. (7), we can obtain exact traveling solutions of Eq. (1):

$$u_3(x,t) = \begin{cases} 3\lambda \sec^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \\ 3\lambda \csc^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \end{cases}, (A > 0, \alpha > 0)$$

$$u_4(x,t) = \begin{cases} 3\lambda - 3\lambda \tanh^2[\sqrt{-\frac{1}{4\alpha}}(x - \lambda t)] \\ 3\lambda - 3\lambda \coth^2[\sqrt{-\frac{1}{4\alpha}}(x - \lambda t)] \end{cases}, (A < 0, \alpha < 0)$$

$$u_5(x,t) = \begin{cases} 2\lambda - 3\lambda \sec^2[\sqrt{\frac{-1}{4\alpha}}(x - \lambda t)] \\ 2\lambda - 3\lambda \csc^2[\sqrt{\frac{-1}{4\alpha}}(x - \lambda t)] \end{cases}, (A > 0, \alpha < 0)$$

$$u_6(x,t) = \begin{cases} -\lambda + 3\lambda \tanh^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \\ -\lambda + 3\lambda \coth^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \end{cases}, (A < 0, \alpha > 0)$$

$$\begin{aligned}
u_7(x, t) &= \begin{cases} 3\lambda + \frac{3\lambda}{A^2} \cot^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \\ 3\lambda + \frac{3\lambda}{A^2} \tan^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \end{cases}, (A > 0, \alpha > 0) \\
u_8(x, t) &= \begin{cases} 3\lambda - \frac{3\lambda}{A^2} \tanh^2[\sqrt{\frac{-1}{4\alpha}}(x - \lambda t)] \\ 3\lambda - \frac{3\lambda}{A^2} \coth^2[\sqrt{\frac{-1}{4\alpha}}(x - \lambda t)] \end{cases}, (A < 0, \alpha < 0) \\
u_9(x, t) &= \begin{cases} -\lambda - \frac{3\lambda}{A^2} \cot^2[\sqrt{\frac{-1}{4\alpha}}(x - \lambda t)] \\ -\lambda - \frac{3\lambda}{A^2} \tan^2[\sqrt{\frac{-1}{4\alpha}}(x - \lambda t)] \end{cases}, (A > 0, \alpha < 0) \\
u_{10}(x, t) &= \begin{cases} -\lambda + \frac{3\lambda}{A^2} \coth^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \\ -\lambda + \frac{3\lambda}{A^2} \tanh^2[\sqrt{\frac{1}{4\alpha}}(x - \lambda t)] \end{cases}, (A < 0, \alpha > 0) \\
u_{11}(x, t) &= \begin{cases} \frac{3}{2}\lambda + \frac{3\lambda}{4} \tan^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] + \frac{3\lambda}{4A^2} \cot^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] \\ \frac{3}{2}\lambda + \frac{3\lambda}{4} \cot^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] + \frac{3\lambda}{4A^2} \tan^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] \end{cases}, (A > 0, \alpha > 0) \\
u_{12}(x, t) &= \begin{cases} \frac{3}{2}\lambda - \frac{3\lambda}{4} \tanh^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] - \frac{3\lambda}{4A^2} \coth^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] \\ \frac{3}{2}\lambda - \frac{3\lambda}{4} \coth^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] - \frac{3\lambda}{4A^2} \tanh^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] \end{cases}, (A < 0, \alpha < 0) \\
u_{13}(x, t) &= \begin{cases} \frac{1}{2}\lambda - \frac{3\lambda}{4} \tan^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] - \frac{3\lambda}{4A^2} \cot^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] \\ \frac{1}{2}\lambda - \frac{3\lambda}{4} \cot^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] - \frac{3\lambda}{4A^2} \tan^2[\sqrt{\frac{-1}{16\alpha}}(x - \lambda t)] \end{cases}, (A > 0, \alpha < 0) \\
u_{14}(x, t) &= \begin{cases} \frac{1}{2}\lambda + \frac{3\lambda}{4} \tanh^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] + \frac{3\lambda}{4A^2} \coth^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] \\ \frac{1}{2}\lambda + \frac{3\lambda}{4} \coth^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] + \frac{3\lambda}{4A^2} \tanh^2[\sqrt{\frac{1}{16\alpha}}(x - \lambda t)] \end{cases}, (A < 0, \alpha > 0)
\end{aligned}$$

Remark: The solutions are equivalent to the solutions given in [9].

Case III $C = 1, A = 0, B \neq 0$:

$$a_0 = 2\lambda, a_1 = \frac{12\lambda}{B}, a_2 = \frac{12\lambda}{B^2}, b_2 = b_1 = 0, k = \sqrt{\frac{1}{B^2\alpha}}, (\alpha > 0)$$

where λ is left as a free parameter.

Substituting these values into Eq. (12) and Eq. (8), we can obtain exact traveling solutions of Eq. (1):

$$\begin{aligned}
u_{15}(x, t) &= 2\lambda + 12B\lambda \{c_0 \exp[-B\sqrt{\frac{1}{B^2\alpha}}(x - \lambda t)] - 1\}^{-1} \\
&\quad + 12\lambda \{c_0 \exp[-B\sqrt{\frac{1}{B^2\alpha}}(x - \lambda t)] - 1\}^{-2}, \quad (\alpha > 0)
\end{aligned}$$

Case IV $C = 1, A \neq 0, B \neq 0$:

$$\begin{aligned}
a_0 = 0, a_1 = -\frac{12B\lambda}{8A+B^2}, a_2 = -\frac{12\lambda}{8A+B^2}, b_2 = b_1 = 0, k = \sqrt{-\frac{1}{(8A+B^2)\alpha}}, B \neq 1, \\
(\alpha < 0, B^2 - 4A > 0)
\end{aligned}$$

$$a_0 = \frac{12A\lambda}{4A-1}, a_1 = a_2 = \frac{12\lambda}{4A-1}, b_2 = b_1 = 0, B = 1, k = \sqrt{\frac{1}{(4A-1)\alpha}}, (\alpha > 0, A > \frac{1}{4})$$

$$\begin{aligned}
a_0 = \frac{4A^2 + 8A}{4A^2 - B^2}\lambda, a_1 = a_2 = 0, b_1 = \frac{12B}{4A^3 - AB^2}\lambda, b_2 = \frac{12}{4A^2 - B^2}\lambda, k = \sqrt{\frac{1}{(4A^2 - B^2)\alpha}}, \\
(B^2 - 4A > 0, (4A^2 - B^2)\alpha > 0)
\end{aligned}$$

$$a_0 = \frac{4A^2 - 8A - 2B^2}{4A^2 - B^2} \lambda, a_1 = a_2 = 0, b_1 = -\frac{12B}{4A^3 - AB^2} \lambda, b_2 = -\frac{12}{4A^2 - B^2} \lambda,$$

$$k = \sqrt{-\frac{1}{(4A^2 - B^2)\alpha}}, \quad (B^2 - 4A > 0, (4A^2 - B^2)\alpha < 0)$$

where λ is left as a free parameter.

Substituting these values into Eq. (12) and Eq. (9), we can obtain exact traveling solutions of Eq. (1):

$$u_{16} = -\frac{12B\lambda}{8A+B^2} \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{-\frac{1}{(8A+B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{-\frac{1}{(8A+B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}^2 \quad (\alpha < 0, B^2 - 4A > 0)$$

$$-\frac{12\lambda}{8A+B^2} \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{-\frac{1}{(8A+B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{-\frac{1}{(8A+B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}$$

$$u_{17} = \frac{12A\lambda}{4A-1} + \frac{12\lambda}{4A-1} \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{\frac{1}{(4A-1)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{\frac{1}{(4A-1)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}^2 \quad (\alpha > 0, A > \frac{1}{4})$$

$$+ \frac{12\lambda}{4A-1} \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{\frac{1}{(4A-1)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{\frac{1}{(4A-1)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}$$

$$u_{18} = \frac{4A^2 + 8A}{4A^2 - B^2} \lambda + \frac{12B}{4A^3 - AB^2} \lambda \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}^{-1}$$

$$+ \frac{12}{4A^2 - B^2} \lambda \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}^{-2} \quad \left(\begin{array}{l} B^2 - 4A > 0 \\ (4A^2 - B^2)\alpha > 0 \end{array} \right)$$

$$u_{19} = \frac{4A^2 - 8A - 2B^2}{4A^2 - B^2} \lambda - \frac{12B}{4A^3 - AB^2} \lambda \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{-\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{-\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}^{-1}$$

$$- \frac{12\lambda}{4A^3 - AB^2} \left\{ \frac{Y_1 - Y_2 C_1 \exp \left[\sqrt{-\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]}{1 - C_1 \exp \left[\sqrt{-\frac{1}{(4A^2 - B^2)\alpha}} (Y_1 - Y_2)(x - \lambda t) \right]} \right\}^{-2}$$

$$(B^2 - 4A > 0, (4A^2 - B^2)\alpha < 0)$$

Remark: As we know, the solutions $u_{16} - u_{19}$ are new solutions, and the solutions are obtained in the paper is very extensive, our method is more brief and direct to find solutions to various solutions of nonlinear evolution equations.

4 Conclusion

In this paper, a polynomial expansion method with a computerized symbolic computation is used for constructing new periodic wave solutions for nonlinear Equal width wave equation. As a result, many exact traveling wave solutions are successfully obtained, and this enriches the literature on the type of wave equations of Equal width wave equation. Of course, this method can also be applied to other nonlinear evolution equations.

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References

- [1] M A Abdou: Exact TravellingWave Solutions in a Nonlinear Elastic Rod Equation. *International Journal of Nonlinear Science*. 7(2):167-173(2009)
- [2] E.J. Parkes, B.R. Duffy:An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations. *Comput. Phys.Commun.* 98:288-296(1996)
- [3] Dianchen Lu, Baojian Hong, Lixin Tian: New solitary wave and periodic wave solutions for general types of KdV and KdVCBurgers equations. *Communications in Nonlinear Science and Numerical Simulation* . 14:77-84(2009)
- [4] K. R. Raslan: The Application of Hes Exp-function Method for MKdV and Burgers Equations with Variable Coefficients. *International Journal of Nonlinear Science*. 7(2):174-181(2009)
- [5] Lixia Wang, Jiangbo Zhou, Lihong Ren: The Exact SolitaryWave Solutions for a Family of BBM Equation. *International Journal of Nonlinear Science*. 1(1):58-64(2006)
- [6] Gamze Tanolu: Hirota Method for Solving Reaction-Diffusion Equations with Generalized Nonlinearity. *International Journal of Nonlinear Science*. 1(1):30-36(2006)
- [7] Liu Jianbin, Yang Kongqing: The extended F-expansion method and exact solutions of nonlinear PDEs. *Chaos,Solitons and Fractals*. 22(1):111-21(2004)
- [8] Abdulkadir Dong: Application of Glerkin's method to equal width wave equation. *Applied Mathematics and Computation*. 160:65-76(2005)
- [9] Zhao Yunmei, Rui Weiguo: All Kinds of the Periodic Solutions of Jacobian Elliptic function Type and the Soliton Solutions for Equal Width Wave Equation. *Journal of Sichuan Normal University (Natural Science)*. 31(2):190-193(2008)