

A Generalized $\left(\frac{G'}{G}\right)$ -expansion Method and Its Application to the MKdV Equation

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Abstract: A generalized $\left(\frac{G'}{G}\right)$ -expansion method is devised to construct the exact traveling wave solutions of mKdV equation. With the aid of symbolic computation, many hyperbolic function solutions, trigonometric function solutions and rational function solutions are obtained. It is shown that the proposed method is more effective and powerful than the $\left(\frac{G'}{G}\right)$ -expansion method in constructing the traveling wave solutions. It can be used for many other nonlinear evolution equations in mathematical physics.

Keywords: generalized $\left(\frac{G'}{G}\right)$ -expansion method; mKdV equation; traveling wave solutions

1 Introduction

The nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and so on.

During the past four decades, many efficient and powerful methods have been developed to find exact solutions of some nonlinear evolution equations(NLEEs). For example, Hirota's bilinear method [1,2], solitary wave ansatz [3], the tanh method [4,5], Backlund transformation [6,7], symmetry method [8,9], the sine-cosine function method [10], the exp function method [11] and so on. The above methods derived many types of solutions from most nonlinear evolution equations. But these methods mentioned above have some limitations in their applications and involve tedious computation. With the development of computation systems like Maple or Mathematica, recently, the searching for exact solutions of NLEEs has attracted much attention.

In recent years, Wang et al.[12] introduced a new method called the $\left(\frac{G'}{G}\right)$ -expansion method, which is derived from the well-known homogeneous balance method and F-expansion method. They obtained many traveling wave solutions of KdV equation, the mKdV equation, the variant Boussinesq equation and the Hirota-Satsuma equation, and these traveling wave solutions are expressed by the hyperbolic function, the trigonometric function and the rational function. Then they [13] discussed the Broer-Kaup equation and the approximate long water wave equation. More recently Li et al.[14] derived the higher-order nonlinear Schrödinger equation solutions by the same method. Bekir A [15] applied the method to Klein-Gordon equation, a symmetric regularized long wave equation and the Drinfeld-Sokolov equation.

The motivation of the present paper is to improve the $\left(\frac{G'}{G}\right)$ -expansion method to the mKdV equation

$$u_t - u^2 u_x + \delta u_{xxx} = 0,$$

which was first proposed as a model to describe the transverse vibration of a string.

The rest of this paper is organized as follows. In Section 2, we give the description of the generalized $\left(\frac{G'}{G}\right)$ -expansion method. In Section 3, we apply this method to mKdV equation. In Section 4, some conclusions are given.

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2 Description of the generalized $\left(\frac{G'}{G}\right)$ -expansion method

In this section, we describe the generalized $\left(\frac{G'}{G}\right)$ -expansion method for finding the traveling wave solutions of NLEEs. Suppose that a nonlinear equation, say in two independent variables x and t , is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots), \quad (1)$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1: We suppose that

$$u(x, t) = u(\xi), \xi = x - Vt, \quad (2)$$

where V is to be determined. Therefore the equation (1) can be reduced to an ordinary equation

$$Q(u, -Vu', u', V^2u'', -Vu'', u'', \dots) = 0. \quad (3)$$

Step 2: Suppose that the solution of ODE (3) can be expressed by a polynomial as follows:

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \beta_m \left(\frac{G'}{G}\right)^{-m} + \dots, \quad (4)$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (5)$$

the coefficient $\alpha_m, \beta_m, \dots, \lambda, \mu$ are constants to be determined in the next step, $\alpha_m \neq 0, \beta_m \neq 0$. The unwritten part in (4) is also a polynomial in $\left(\frac{G'}{G}\right)$. The positive integer m will be determined by the homogeneous balance method between the highest order derivative term and highest order nonlinear term appearing in ODE (3).

Step 3: By substituting (4) into ODE (3) and using second order LODE (5), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq.(3) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, we obtain a set of algebraic equations for $\alpha_m, \beta_m, \dots, \lambda$ and μ .

Step 4: Assume that the constants $\alpha_m, \beta_m, \dots, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of ODE (4) are well-known for us, then substituting α_m, β_m, \dots and general solutions of (5) into (4), we have more traveling wave solutions of the nonlinear evolution equation (1).

3 Application to the MKdV equation

$$u_t - u^2u_x + \delta u_{xxx} = 0, \quad \delta > 0 \quad (6)$$

Suppose $u(x, t) = u(\xi), \xi = x - Vt$, where the speed V is to be determined later.

$$-Vu' - u^2u' + \delta u''' = 0. \quad (7)$$

Integrating it with respect to ξ once, it yields

$$C - Vu - \frac{1}{3}u^3 + \delta u'' = 0, \quad (8)$$

where C is an integration constant that is to be determined later.

Considering the homogeneous balance between u'' and u^3 in Eq.(8), we get $m = 1$, so we can suppose the solution of Eq.(6) is of the form

$$u(\xi) = b_1 \left(\frac{G'}{G}\right) + b_2 \left(\frac{G'}{G}\right) + b_0, \quad (9)$$

where b_i is to be determined later, $G = G(\xi)$ satisfies the second order LODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0. \quad (10)$$

By using Eq.(9) and (10), it is derived that

$$u''(\xi) = \mu b_1 \lambda + b_2 \lambda + (b_1 \lambda^2 + 2b_1 \mu) \left(\frac{G'}{G}\right) + 3b_1 \lambda \left(\frac{G'}{G}\right)^2 + 2b_1 \left(\frac{G'}{G}\right)^3 \\ + (2b_2 \mu + b_2 \lambda^2) \left(\frac{G}{G'}\right) + 3b_2 \mu \lambda \left(\frac{G}{G'}\right)^2 + 2b_2 \mu^2 \left(\frac{G}{G'}\right)^3, \quad (11)$$

$$u^3(\xi) = b_0^3 + 6b_1 b_2 b_0 + (3b_1 b_0^2 + 3b_2 b_1^2) \left(\frac{G'}{G}\right) + 3b_0 b_1^2 \left(\frac{G'}{G}\right)^2 + b_1^3 \left(\frac{G'}{G}\right)^3 \\ + (3b_2 b_0^2 + 3b_1 b_2^2) \left(\frac{G}{G'}\right) + 3b_0 b_2^2 \left(\frac{G}{G'}\right)^2 + b_2^3 \left(\frac{G}{G'}\right)^3. \quad (12)$$

By substituting (9), (11) and (12) into Eq.(8) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ and $\left(\frac{G}{G'}\right)$ together, the left-hand side of Eq.(8) is converted into another polynomial in $\left(\frac{G'}{G}\right)$ and $\left(\frac{G}{G'}\right)$. Equating each coefficient of this polynomial to zero, we can get the following equations:

$$C - V b_0 - \frac{1}{3} b_0^3 - 2b_1 b_2 b_0 + \delta b_1 \lambda \mu + \delta b_2 \lambda = 0, \\ -V b_1 - b_1 b_0^2 - b_2 b_1^2 + \delta b_1 \lambda^2 + 2\delta b_1 \mu = 0, \\ -V b_2 - b_2 b_0^2 - b_1 b_2^2 + \delta b_2 \lambda^2 + 2\delta b_2 \mu = 0, \\ -b_1^2 b_0 + 3\delta b_1 \lambda = 0, \\ -b_2^2 b_0 + 3\delta b_2 \lambda \mu = 0, \\ -\frac{1}{3} b_1^3 + 2\delta b_1 = 0, \\ -\frac{1}{3} b_2^3 + 2\delta b_2 \mu^2 = 0.$$

Solving the algebraic equation above by Mathematica, it yields

First solution set:

$$b_2 = 0, \quad b_1 = \pm\sqrt{6\delta}, \quad b_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, \quad V = \frac{1}{2}(-\delta\lambda^2 + 4\delta\mu), \quad C = 0. \quad (13)$$

Second solution set:

$$b_2 = \pm\sqrt{6\delta}, \quad b_1 = 0, \quad b_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, \quad V = \frac{1}{2}(-\delta\lambda^2 + 4\delta\mu), \quad C = 0. \quad (14)$$

Third solution set:

$$b_2 = \pm\sqrt{6\delta}\mu, \quad b_1 = \pm\sqrt{6\delta}, \quad b_0 = \pm\frac{1}{2}\lambda\sqrt{6\delta}, \quad V = \frac{1}{2}(-\delta\lambda^2 - 8\delta\mu), \quad C = \pm 2\sqrt{6}\lambda\mu\delta^{3/2}. \quad (15)$$

1) Substituting the solution set (13) into (9), (9) can be written as

$$u(\xi) = \pm\sqrt{6\delta} \left(\frac{G'}{G}\right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \quad (16)$$

where $\xi = x - \frac{1}{2}(4\delta\mu - \delta\lambda^2)t$, $C = 0$.

Substituting the general solutions of Eq. (10) into (16), we have three types of traveling wave solutions of the mKdV equation (6), which is the same as the solutions obtained by Wang et al. in[12].

2) Substituting the solution set (14) into (9), (9) can be written as

$$u(\xi) = \pm\sqrt{6\delta} \left(\frac{G}{G'}\right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \quad (17)$$

where $\xi = x - \frac{1}{2}(4\delta\mu - \delta\lambda^2)t, C = 0$.

Substituting the general solutions of Eq. (10) into (17), we have three types of traveling wave solutions of the mKdV equation (6) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solutions:

$$u_{1,2}(\xi) = \pm\sqrt{6\delta} \left(\frac{C_2 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_1 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{(C_1\sqrt{\lambda^2-4\mu} - \lambda C_2) \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + (C_2\sqrt{\lambda^2-4\mu} - \lambda C_1) \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi} \right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \tag{18}$$

where $\xi = x - \frac{1}{2}(4\delta\mu - \delta\lambda^2)t, C_1$ and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$u_{3,4}(\xi) = \pm\sqrt{6\delta} \left(\frac{C_2 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi - C_1 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{(C_1\sqrt{4\mu-\lambda^2} - \lambda C_2) \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + (C_2\sqrt{4\mu-\lambda^2} - \lambda C_1) \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi} \right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \tag{19}$$

where $\xi = x - \frac{1}{2}(4\delta\mu - \delta\lambda^2)t, C_1$ and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_{5,6}(\xi) = \pm\sqrt{6\delta} \left(\frac{2(C_1 + C_2\xi)}{2C_2 - C_1\lambda - C_2\lambda\xi} \right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \tag{20}$$

which is independent of variable t, C_1 and C_2 are arbitrary constants.

3) Substituting the solution set (15) into (9), (9) can be written as

$$u(\xi) = \pm\sqrt{6\delta} \left(\frac{G'}{G} \right) \pm \sqrt{6\delta}\mu \left(\frac{G}{G'} \right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \tag{21}$$

where $\xi = x + \frac{1}{2}(\delta\lambda^2 + 8\delta\mu)t, C = \pm 2\sqrt{6}\lambda\mu\delta^{3/2}$.

Substituting the general solutions of Eq. (10) into (21), we have three types of traveling wave solutions of the mKdV equation (6) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solutions:

$$u_{7,8}(\xi) = \pm\sqrt{6\delta} \left(\frac{C_1^2\lambda \exp 2\sqrt{\lambda^2-4\mu}\xi(-\lambda + \sqrt{\lambda^2-4\mu}) - C_2^2\lambda(\lambda + \sqrt{\lambda^2-4\mu}) - 8C_1C_2\mu \exp \sqrt{\lambda^2-4\mu}\xi}{(C_2 + C_1 \exp \xi\sqrt{\lambda^2-4\mu})[-C_1 \exp \sqrt{\lambda^2-4\mu}\xi(-\lambda + \sqrt{\lambda^2-4\mu}) + C_2(\lambda + \sqrt{\lambda^2-4\mu})]} \right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \tag{22}$$

where $\xi = x + \frac{1}{2}(\delta\lambda^2 + 8\delta\mu)t, C = \pm 2\sqrt{6}\lambda\mu\delta^{3/2}, C_1$ and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$u_{9,10}(\xi) = \pm\sqrt{6\delta} \left(\frac{C_1^2\lambda \exp 2i\sqrt{4\mu-\lambda^2}\xi(\sqrt{4\mu-\lambda^2} - \lambda) - C_2^2\lambda(\lambda + i\sqrt{4\mu-\lambda^2}) - 8C_1C_2\mu \exp i\xi\sqrt{4\mu-\lambda^2}}{(C_2 + C_1 \exp i\xi\sqrt{4\mu-\lambda^2})[C_1 \exp i\xi\sqrt{4\mu-\lambda^2}(\lambda - i\sqrt{4\mu-\lambda^2}) + C_2(\lambda + i\sqrt{4\mu-\lambda^2})]} \right) \pm \frac{1}{2}\lambda\sqrt{6\delta}, \tag{23}$$

When $\lambda^2 - 4\mu = 0$,

$$u_{11,12}(\xi) = \pm\sqrt{6\delta} \left(\frac{C_2}{C_1 + C_2\xi} \right) \pm \sqrt{6\delta}\mu \left(\frac{2(C_1 + C_2\xi)}{2C_2 - C_1\lambda - C_2\lambda\xi} \right), \tag{24}$$

which is independent of variable t, C_1 and C_2 are arbitrary constants.

Remark 1 Many exact traveling wave solutions of the mKdV equation are successfully obtained by the generalized $\left(\frac{G'}{G}\right)$ -expansion method. All solutions of mKdV equation in [12] can be obtained by using the generalized $\left(\frac{G'}{G}\right)$ -expansion method. But, the solutions (18)-(20) and (22)-(24) had not been given in Ref.[12]. It shows that the proposed method is more powerful in constructing exact solutions of NLEEs than the $\left(\frac{G'}{G}\right)$ -expansion method.

4 Conclusion

In this paper, the generalized $\left(\frac{G'}{G}\right)$ -expansion method is proposed to obtain more exact solutions of mKdV equation. As a result, many traveling wave solutions are obtained including hyperbolic function solutions, trigonometric function solutions and rational function solutions. The proposed method is more effective and powerful than the $\left(\frac{G'}{G}\right)$ -expansion method. The generalized $\left(\frac{G'}{G}\right)$ -expansion method can be used for many other NLEEs in mathematical physics.

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