

Travelling Wave Solutions for the Gilson-Pickering Equation by Using the Simplified G'/G -expansion Method

Xinghua Fan *, Shouxiang Yang, Dan Zhao

Department of Mathematics, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R.China

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Abstract: We simplify the expression of G'/G in the G'/G expansion method to tanh, coth, cot and rational forms under some conditions. The simplified G'/G -method can be thought as a combination of tanh-coth method and cot method. Travelling wave solutions for the Gilson-Pickering equation are considered. The equation contains the Fornberg-Whitham equation, the Rosenau-Hyman equation and the Fuchssteiner-Fokas-Camassa-Holm equation as its special cases. We construct its travelling wave solutions involving parameters by using the simplified G'/G expansion method. The resulting solutions are expressed by hyperbolic, trigonometric functions and rational functions.

Keywords: the G'/G -expansion method; traveling wave solutions; solitary wave solutions; homogeneous balance; hyperbolic function solutions; trigonometric function solutions

1 Introduction

We consider the class of fully nonlinear third-order partial differential equations (PDEs)

$$u_t - \epsilon u_{xxt} + 2\kappa u_x - uu_{xxx} - \alpha uu_x - \beta u_x u_{xx} = 0, \quad (1)$$

where ϵ, κ, α and β are arbitrary constants. The equation was introduced by Gilson and Pickering (GP) in [11] and was called the Gilson-Pickering equation in [3] where solutions for $\beta = 1$ were studied.

Our results are corresponding to $\beta = -2$. Three special cases of Eq.(1) have appeared in the literature. Up to some rescalings, these are: the Fornberg-Whitham (FW) equation [9, 20, 21], the Rosenau-Hyman (RH) equation [15], the Fuchssteiner-Fokas-Camassa-Holm (CH) equation [2, 10].

Studies to the GP equation (1) include [4, 11]. For $\beta = 1$, the qualitative behavior and exact travelling wave solutions of Eq.(1) were studied by using the qualitative theory of polynomial differential system [3]. We are interested to find the explicit solutions of Eq. (1).

With the fast development of symbolic computation systems, directly searching for exact solutions of PDEs by symbolic computation has attracted much attention. Various methods have appeared such as the homogenous balance method [16], the Sine-Cosine method [8], the Jacobi elliptic function method [7], the sine-gordon equation expansion method [24], the F-expansion method [6], the Exp-function method [12, 13, 22, 23] and so on.

In the present paper, we shall use the G'/G -expansion method [17]. It is based on the explicit linearization of nonlinear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. The G'/G -expansion method have been applied to many kinds of PDEs [1, 5, 18]. It is also applied to lattice equation in [25].

It is pointed out in [14] that the G'/G -expansion method coincides with the truncated expansion method if we use the travelling wave solutions. The G'/G -expansion method is equivalent to application of the simplest equation method with the Riccati equation, to the tanh-function method and to the truncated expansion method.

*Corresponding author. E-mail address: fan131@ujs.edu.cn

We simplify the expression of G'/G in the G'/G expansion method to tanh, coth, cot and rational forms under some conditions and call it the simplified G'/G method. We say that the simplified G'/G -method can be thought as a combination of tanh-coth method and cot method. Hyperbolic, trigonometric and rational function solutions can be found.

The layout of this paper is organized as follows. In Section 2, we give the description of the simplified G'/G -expansion method. In Section 3, we apply this method to the GP equation. Conclusions are given in the last section.

2 Description of the (G'/G) -expansion method

We first describe the G'/G -expansion method [17] with two independent variables. For a given nonlinear PDE

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \tag{2}$$

where $u = u(x, t)$ is an unknown function, P is a polynomial in u and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. There are mainly four steps in the G'/G -expansion method.

Step 1. The travelling wave variable

$$u(x, t) = \phi(\xi), \quad \xi = x - Vt, \tag{3}$$

permits us to reduce Eq. (2) to an ODE for $u = \phi(\xi)$ in the form

$$P(\phi, -V\phi', \phi', V^2\phi'', -V\phi'', \phi'', \dots) = 0, \tag{4}$$

where “'” denotes derivative about ξ .

Step 2. Suppose that the solution of ODE (4) can be expressed by a polynomial in G'/G as follows:

$$\phi(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G}\right)^i, \tag{5}$$

where $G = G(\xi)$ satisfies the following second order linear ODE

$$G'' + \lambda G' + \mu G = 0, \tag{6}$$

where a_i ($i = 0, 1, 2, \dots, m$), λ and μ are constants to be determined later, $a_m \neq 0$. The degree of the polynomial m can be determined by balancing the highest order derivative with nonlinear terms.

Step 3. Substituting (5) into (4) and using the second order linear ODE (6) and then equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations with respect to a_i ($i = 0, 1, 2, \dots, m$), V , λ , and μ . Solving the algebraic system, we may find the values of unknowns.

Step 4. Substituting a_i ($i = 0, 1, 2, \dots, m$), V , λ , μ gotten in Step 3 and the general solutions of Eq. (6) into (5) we can obtain more traveling wave solutions of the nonlinear PDE (2). Solutions to Eq.(??) depending on whether $\lambda^2 - 4\mu > 0$, $\lambda^2 - 4\mu < 0$, $\lambda^2 - 4\mu = 0$,

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{C_1 \sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + C_2 \cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2} \frac{C_1 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi - C_2 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi}{C_1 \sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi + C_2 \cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\right)\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \\ \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0, \end{cases} \tag{7}$$

As a result, solutions in sinh-cosh, or sin-cos forms are given in [17] and others.

Next we will give the simplified method.

As pointed in [14], some authors do not simplify the solutions of differential equations. We find that these form should be simplified to

$$\frac{G'}{G} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2} \tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \tanh \xi_0 = \frac{C_2}{C_1}, \left|\frac{C_2}{C_1}\right| > 1, \\ \frac{\sqrt{\lambda^2-4\mu}}{2} \coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \xi_0\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \coth \xi_0 = \frac{C_2}{C_1}, \left|\frac{C_2}{C_1}\right| < 1, \\ \frac{\sqrt{4\mu-\lambda^2}}{2} \cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \xi_0\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \tan \xi_0 = \frac{C_2}{C_1} \\ \frac{C_2}{C_1+C_2\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \end{cases} \quad (8)$$

Simplifying (7) with (8), solutions can be written in more strait form. Now we call the method the simplified G'/G -method. We will use (8) to get travelling wave solution of Eq.(1).

3 Application to the Gilson-Pickering equation

In this section, we apply the simplified G'/G -expansion method to construct the traveling wave solutions of the Gilson-Pickering equation (1).

Combining the independent variables x and t into one variable $\xi = x - Vt$, we suppose that

$$u(x, t) = \phi(\xi), \xi = x - Vt. \quad (9)$$

Then Eq. (1) is converted into an ODE for $u = \phi(\xi)$

$$(2\kappa - V)\phi' + \epsilon V\phi''' - \phi\phi''' - \alpha\phi\phi' - \beta\phi'\phi'' = 0. \quad (10)$$

Integrating Eq. (10) with respect to ξ once yields

$$(2\kappa - V)\phi + \epsilon V\phi'' - \phi\phi'' - \frac{\alpha}{2}\phi^2 + \frac{1-\beta}{2}(\phi')^2 = C \quad (11)$$

where C is an integration constant that is to be determined later.

Suppose that the solution of the ODE (10) can be expressed by polynomials in terms of G'/G as follows:

$$\phi(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G}\right)^i, \quad (12)$$

while $G = G(\xi)$ satisfies (6).

The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (11). To determine the degree of the polynomial solutions, we can only substitute the leading term. Let the degree in G'/G of (12) be $D(\phi)$. Then $D(\phi^p \left(\frac{d^s \phi}{d\xi^s}\right)^q) = pD(\phi) + (s + D(\phi)q)$. So we get $m = 2$ and

$$\phi(\xi) = a_2 \left(\frac{G'}{G}\right)^2 + a_1 \left(\frac{G'}{G}\right) + a_0, \quad (13)$$

By substituting (13) and its derivatives into Eq. (11) and collecting all terms with the same power of G'/G together, the left-hand side of Eq. (11) is converted into another polynomial in G'/G . Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $a_2, a_1, a_0, V, \lambda, \mu$ and C . For the length of the paper, we just omit it. We can see that only when $\beta = -2$ can we get some solutions. In the rest text we will take $\beta = -2$. Solving the algebraic equations, yields three sets of solutions:

$$\begin{aligned} \text{Set A: } a_2 &= 4 \frac{a_0 + \alpha \epsilon a_0 - 2 \kappa \epsilon}{(1 + \alpha \epsilon)(\lambda^2 - \alpha)}, a_1 = a_2 \lambda, a_0 = a_0, V = \frac{2 \kappa}{1 + \alpha \epsilon}, \\ C &= 2\alpha \left(\frac{\kappa \epsilon}{1 + \alpha \epsilon}\right)^2, \mu = \frac{\lambda^2 - \alpha}{4}, \end{aligned} \quad (14)$$

$$\begin{aligned} \text{Set B: } a_2 &= 12 \frac{a_0 + \alpha \epsilon a_0 - 2 \kappa \epsilon}{(1 + \alpha \epsilon)(3 \lambda^2 + \alpha)}, a_1 = a_2 \lambda, a_0 = a_0, V = \frac{2 \kappa}{1 + \alpha \epsilon}, \\ C &= 2 \alpha \frac{(\alpha a_0(1 + \alpha \epsilon) + \epsilon \kappa(3 \lambda^2 - \alpha))(3 \epsilon \kappa(\lambda^2 + \alpha) - \alpha a_0(1 + \alpha \epsilon))}{(1 + \alpha \epsilon)^2(3 \lambda^2 + \alpha)^2}, \\ \mu &= \frac{\lambda^2 + \alpha}{4}, \end{aligned} \tag{15}$$

$$\begin{aligned} \text{Set C: } a_2 &= 4 \frac{\epsilon(2 \epsilon \kappa - \alpha \epsilon a_0 - a_0)}{(1 - \lambda^2 \epsilon)(1 + \alpha \epsilon)}, a_1 = \lambda a_2, a_0 = a_0, \\ V &= -\frac{\alpha^2 \epsilon^2 a_0 - 2 \epsilon^2 \kappa \alpha + 6 \epsilon^2 \lambda^2 \kappa - 4 \epsilon \kappa - a_0}{3 \epsilon(1 - \lambda^2 \epsilon)(1 + \alpha \epsilon)} \\ C &= -\frac{2 \kappa(-\alpha^2 \epsilon^2 a_0 + a_0 - 2 \epsilon \kappa - \epsilon^2 \kappa \alpha + 3 \epsilon^3 \kappa \alpha \lambda^2)}{3(1 - \lambda^2 \epsilon)(1 + \alpha \epsilon)^2}, \mu = \frac{\lambda^2 \epsilon - 1}{4 \epsilon}. \end{aligned} \tag{16}$$

If the integration constant C in Eq.(11) is correspondingly taken as that in (14), (15) and (16), we can write (13) in expression of G'/G and the parameters $\epsilon, \kappa, \alpha, \beta$. Substituting (8) into it we can derive the travelling wave solutions of Eq. (1).

3.1 First travelling solution set

To Set A, substituting (14) and (8) into (12), we deduce the following three types of traveling wave solutions of the GP equation (1):

Case 1-1. When $\lambda^2 - 4\mu = \alpha > 0$, we have the hyperbolic function travelling wave solution

$$u_1^+(x, t) = \frac{(-\alpha \epsilon a_0 + 2 \epsilon \kappa - a_0) \alpha}{(1 + \alpha \epsilon)(\lambda^2 - \alpha)} \operatorname{sech}^2\left(\frac{\sqrt{\alpha}}{2} \xi + \xi_0\right) + \frac{2 \epsilon \kappa}{1 + \alpha \epsilon} \tag{17}$$

where $\xi = x - \frac{2 \kappa}{1 + \alpha \epsilon} t, \xi_0 = \tanh^{-1} \frac{C_2}{C_1}, |C_2/C_1| > 1$, and

$$u_2^+(x, t) = \frac{(\alpha \epsilon a_0 - 2 \epsilon \kappa + a_0) \alpha}{(1 + \alpha \epsilon)(\lambda^2 - \alpha)} \operatorname{csch}^2\left(\frac{\sqrt{\alpha}}{2} \xi + \xi_0\right) + \frac{2 \epsilon \kappa}{1 + \alpha \epsilon} \tag{18}$$

where $\xi = x - \frac{2 \kappa}{1 + \alpha \epsilon} t, \xi_0 = \coth^{-1} \frac{C_2}{C_1}, |C_2/C_1| < 1$,

Solution (17) is a well-known bounded solitary wave solution.

Case 1-2. When $\lambda^2 - 4\mu = \alpha < 0$, we have the trigonometric solutions

$$u_1^-(x, t) = \frac{(a_0 + \alpha \epsilon a_0 - 2 \epsilon \kappa) \alpha}{(1 + \alpha \epsilon)(\lambda^2 - \alpha)} \cot^2\left(\frac{\sqrt{-\alpha}}{2} \xi + \xi_0\right) - \frac{(a_0 + \alpha \epsilon a_0 - 2 \epsilon \kappa) \lambda^2}{(1 + \alpha \epsilon)(\lambda^2 - \alpha)} + a_0, \tag{19}$$

where $\xi = x - \frac{2 \kappa}{1 + \alpha \epsilon} t, \xi_0 = \tan^{-1} \frac{C_2}{C_1}$.

Case 1-3. When $\lambda^2 - 4\mu = \alpha = 0$, the rational solutions can be found:

$$u_1^0(x, t) = 4 \frac{a_0 - 2 \epsilon \kappa}{\lambda^2} \frac{C_2^2}{(C_1 + C_2 \xi)^2} - a_0 + 2 \epsilon \kappa \tag{20}$$

where $\xi = x - 2 \kappa t, C_1$ and C_2 are arbitrary constants. The integration constants C in (11) is now zero.

We can see that all travelling wave solutions are independent of variable t when $\kappa = 0$.

3.2 Second travelling solution set

Substituting (15) and (8) into (12), three types of traveling wave solutions of the GP equation (1) are obtained:

Case 2-1. When $\lambda^2 - 4\mu = -\alpha > 0$. If $|C_2/C_1| > 1$, we get a solitary solution

$$u_3^+(x, t) = 3 \frac{(-\alpha \epsilon a_0 + 2 \epsilon \kappa - a_0) \alpha}{(1 + \alpha \epsilon) (3 \lambda^2 + \alpha)} \operatorname{sech}^2 \left(\frac{\sqrt{-\alpha}}{2} \xi + \xi_0 \right) + 2 \frac{2 \alpha^2 \epsilon a_0 - 3 \alpha \epsilon \kappa + 2 \alpha a_0 + 3 \epsilon \lambda^2 \kappa}{(1 + \alpha \epsilon) (3 \lambda^2 + \alpha)}, \quad (21)$$

where $\xi = x - \frac{2\kappa}{1+\alpha\epsilon}t$, $\xi_0 = \tanh^{-1} \frac{C_2}{C_1}$.

If $|C_2/C_1| < 1$, letting $C_2/C_1 = \coth \xi_0$, we have

$$u_4^+(x, t) = 3 \frac{(\alpha \epsilon a_0 - 2 \epsilon \kappa + a_0) \alpha}{(1 + \alpha \epsilon) (3 \lambda^2 + \alpha)} \operatorname{csch}^2 \left(\frac{\sqrt{-\alpha}}{2} \xi + \xi_0 \right) + 2 \frac{2 \alpha^2 \epsilon a_0 - 3 \alpha \epsilon \kappa + 2 \alpha a_0 + 3 \epsilon \lambda^2 \kappa}{(1 + \alpha \epsilon) (3 \lambda^2 + \alpha)}. \quad (22)$$

where $\xi = x - \frac{2\kappa}{1+\alpha\epsilon}t$, $\xi_0 = \coth^{-1} \frac{C_2}{C_1}$,

Case 2-2. When $\lambda^2 - 4\mu = -\alpha < 0$, we can get a periodic solution

$$u_2^-(x, t) = 3 \alpha \frac{(\alpha \epsilon a_0 - 2 \epsilon \kappa + a_0)}{(1 + \alpha \epsilon) (3 \lambda^2 + \alpha)} \cot^2 \left(\frac{\sqrt{-\alpha}}{2} \xi + \xi_0 \right)^2 - 3 \frac{(\alpha \epsilon a_0 - 2 \epsilon \kappa + a_0) \lambda^2}{(1 + \alpha \epsilon) (3 \lambda^2 + \alpha)} + a_0, \quad (23)$$

where $\xi = x - \frac{2\kappa}{1+\alpha\epsilon}t$, $\xi_0 = \tan^{-1} \frac{C_2}{C_1}$.

When $\lambda^2 - 4\mu = -\alpha = 0$, we have rational solution just the same as (20).

3.3 Third travelling solution set

Substituting (16) and (8) into (12), we deduce the following traveling wave solutions of the GP equation (1):

Case 3-1: When $\lambda^2 - 4\mu = \frac{1}{\epsilon} > 0$, we can get a solitary solution

$$u_5^+(x, t) = \frac{\epsilon (\alpha \epsilon a_0 - 2 \epsilon \kappa + a_0) \alpha}{(-1 + \lambda^2 \epsilon) (1 - \alpha \epsilon)} \operatorname{sech}^2 \left(\frac{\xi}{2\sqrt{\epsilon}} + \xi_0 \right) - \frac{\alpha^2 \epsilon^2 a_0 - 2 \epsilon^2 \kappa \alpha + 2 \epsilon^2 \lambda^2 \kappa - a_0}{(1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} \quad (24)$$

where $\xi = x + \frac{\alpha^2 \epsilon^2 a_0 - 2 \epsilon^2 \kappa \alpha + 6 \epsilon^2 \lambda^2 \kappa - 4 \epsilon \kappa - a_0}{3 \epsilon (1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} t$, $\xi_0 = \tanh^{-1} \frac{C_2}{C_1}$, $|C_2/C_1| > 1$, and

$$u_6^+(x, t) = \frac{\epsilon (\alpha \epsilon a_0 - 2 \epsilon \kappa + a_0) \alpha}{(1 - \lambda^2 \epsilon) (1 - \alpha \epsilon)} \operatorname{csch}^2 \left(\frac{\xi}{2\sqrt{\epsilon}} + \xi_0 \right) - \frac{\alpha^2 \epsilon^2 a_0 - 2 \epsilon^2 \kappa \alpha + 2 \epsilon^2 \lambda^2 \kappa - a_0}{(1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} \quad (25)$$

where $\xi = x + \frac{\alpha^2 \epsilon^2 a_0 - 2 \epsilon^2 \kappa \alpha + 6 \epsilon^2 \lambda^2 \kappa - 4 \epsilon \kappa - a_0}{3 \epsilon (1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} t$, $\xi_0 = \coth^{-1} \frac{C_2}{C_1}$, $|C_2/C_1| < 1$.

Case 3-2: When $\lambda^2 - 4\mu = \frac{1}{\epsilon} < 0$, we have

$$u_3^- = \frac{\epsilon (2 \epsilon \kappa - \alpha \epsilon a_0 - a_0) \alpha}{(1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} \cot^2 \left(\frac{1}{2\sqrt{-\epsilon}} \xi + \xi_0 \right) - \frac{\epsilon (2 \epsilon \kappa - \alpha \epsilon a_0 - a_0) \lambda^2}{(1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} + a_0 \quad (26)$$

where $\xi = x + \frac{\alpha^2 \epsilon^2 a_0 - 2 \epsilon^2 \kappa \alpha + 6 \epsilon^2 \lambda^2 \kappa - 4 \epsilon \kappa - a_0}{3 \epsilon (1 - \lambda^2 \epsilon) (1 + \alpha \epsilon)} t$, $\xi_0 = \tan^{-1} \frac{C_2}{C_1}$.

There is no rational solution for Set C.

4 Conclusions

We successfully obtained exact and explicit analytic solutions with arbitrary parameters to fully nonlinear third-order Gilson-Pickering equation via the simplified G'/G -expansion method. The procedure is simple, direct and constructive with the help of a computer algebra system. These travelling wave solutions are expressed by the hyperbolic functions, trigonometric functions and rational functions. We foresee that our results can be found potentially useful for applications in mathematical physics and applied mathematics including numerical simulation.

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