

Existence of Solutions for Double Perturbed Neutral Functional Evolution Equation

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Abstract: This paper is concerned with double perturbed neutral functional evolution equations with infinite delay in Banach spaces. The existence of mild solutions to such equations is obtained by using the theory of the Hausdorff measure of noncompactness and Darbo fixed point theorem, without the compactness assumption on associated evolution system. Our results improve and generalize some previous results.

Keywords: evolution equations with infinite delay; evolution system; mild solution; Hausdorff measure of noncompactness; phase space.

1 Introduction

In this paper, we discuss double perturbed neutral functional evolution equation with infinite delay

$$\frac{d}{dt}(x(t) - h(t, x_t)) = A(t)x(t) + f(t, x_t) + g(t, x_t), \quad t \in J = [0, b] \quad (1.1)$$

$$x_0 = \varphi \in \mathcal{B} \quad (1.2)$$

where $\{A(t) : t \geq 0\}$ is a family of linear closed operators in a real Banach space X that generates an evolution system $\{U(t, s) : 0 \leq s \leq t < \infty\}$ and $D(A(t)) \subseteq X$ is dense in X . The history $x_t : (-\infty, 0] \rightarrow X$, $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space \mathcal{B} defined axiomatically; g, f, h are appropriate functions.

Differential equations with delay was initialed about existence and stability by Travis and Webb [1] and Webb [2]. Since such equations are often more realistic to describe natural phenomena than those without delay, they have been investigated in variant aspects by many authors. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades, see, for example, [3–6] and references therein. In [7, 8] and [9], the authors studied double perturbed differential equations in Frechet spaces, finite dimensional spaces and Banach space, respectively.

In this paper, we investigate the existence of mild solutions for double perturbed neutral functional evolution equations in Banach spaces (1.1)-(1.2). By using the tools involving the measure of noncompactness and fixed point theory, we obtain existence of mild solution of such equation without the assumption of compactness on the associated evolution system. Our results extend and improve the correspondence results in some previous papers.

2 Preliminaries

In this section we introduce some definitions, notations and preliminary facts which will be used in this paper.

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We employ an axiomatic definition of the phase space \mathcal{B} which is similar to that introduced by Hale and Kato [10] and it is appropriated to treat neutral functional evolution equations.

Definition 1 ([10]) *The phase space \mathcal{B} is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, satisfying the following axioms:*

(A) *If $x : (-\infty, \sigma + b] \rightarrow X$, $b > 0$, such that $x_{\sigma} \in \mathcal{B}$ and $x|_{[\sigma, \sigma+b]} \in C([\sigma, \sigma + b]; X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:*

(i) x_t is in \mathcal{B} ,

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t + \sigma)\|x_{\sigma}\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \rightarrow [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) *For the function x in (A), the mapping $t \rightarrow x_t$ is continuous for $t \in [\sigma, \sigma + b]$.*

(B) *The space \mathcal{B} is complete.*

Definition 2 ([11]) *The Hausdorff's measure of noncompactness χ_Y defined by $\chi_Y(B) = \inf\{r > 0, B$ can be covered by finite number of balls with radii $r\}$, for bounded set B in a Banach space Y .*

Lemma 1 ([11]) *Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:*

(1) B is pre-compact if and only if $\chi_Y(B) = 0$;

(2) $\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(\text{conv}B)$, where \overline{B} and $\text{conv}B$ are the closure convex hull of B respectively;

(3) $\chi_Y(B) \leq \chi_Y(C)$ when $B \subseteq C$;

(4) $\chi_Y(B + C) \leq \chi_Y(B) + \chi_Y(C)$ where $B + C = \{x + y : x \in B, y \in C\}$;

(5) $\chi_Y(B \cup C) = \max\{\chi_Y(B), \chi_Y(C)\}$;

(6) $\chi_Y(\lambda B) = |\lambda|\chi_Y(B)$ for any $\lambda \in \mathbb{R}$;

(7) *If the map $Q : D(Q) \subseteq Y \rightarrow Z$ is Lipschitz continuous with constant k then $\chi_Z(QB) \leq k\chi_Y(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space;*

(8) *If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of Y and $\lim_{n \rightarrow \infty} \chi_Y(W_n) = 0$, then $\bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y .*

Definition 3 ([12]) *The map $Q : W \subseteq Y \rightarrow Y$ is said to be a χ_Y - contraction if there exists a positive constant $k < 1$ such that $\chi_Y(Q(C)) \leq k\chi_Y(C)$ for any bounded close subset $C \subseteq W$, where Y is a Banach space.*

Lemma 2 ([13])(Darbo) *Let $W \subseteq Y$ be a closed convex set and $0 \in W$, $Q : W \rightarrow W$ be continuous and χ_Y - contraction. If the set $\{x \in W : x = \lambda Qx\}$ is bounded for $0 < \lambda < 1$, then there exists at least one fixed point of the map Q in W .*

In this paper we denote χ the Hausdorff's measure of noncompactness of X , χ_C the Hausdorff's measure of noncompactness of $C([0, b]; X)$. To discuss the existence we also need the following lemma.

Lemma 3 ([12]) (1) *If $W \subset C([a, b]; X)$ is bounded, then $\chi(W(t)) \leq \chi_C(W)$, for any $t \in [a, b]$, where $W(t) = \{u(t) : u \in W\} \subseteq X$;*

(2) *If $W \subset C([a, b]; X)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous for $t \in [a, b]$, and*

$$\chi_C(W) = \sup\{\chi(W(t)), t \in [a, b]\};$$

(3) *If $W \subset C([a, b]; X)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous for $t \in [a, b]$, and*

$$\chi\left(\int_a^t W(s)ds\right) \leq \int_a^t \chi W(s)ds,$$

for all $t \in [a, b]$, where $\int_a^t W(s)ds = \left\{\int_a^t x(s)ds : x \in W\right\}$.

Let $\{A(t) : t \geq 0\}$ be a family of closed operators from a real Banach space X into itself and $D(A(t)) = D$ is independent of t .

Definition 4 ([7]) The family of linear bounded operators $\{U(t, s) : 0 \leq s \leq t < \infty\}$ on X is called an evolution system, if the follow properties are satisfied:

- (i) $U(t, t) = I$ where I is the identity operators in X ;
- (ii) $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t < +\infty$;
- (iii) $U(t, s) \in B(X)$ the space of bounded linear operator on X , where for every $(t, s) \in \{(t, s) : 0 \leq s \leq t < +\infty\}$ and for each $x \in X$, the mapping $(t, s) \rightarrow U(t, s)x$ continuous.

Definition 5 The evolution system $\{U(t, s) : 0 \leq s \leq t < \infty\}$ is said to be equicontinuous if for each bounded subset B in X , $\{s \rightarrow U(t, s)x : x \in B\}$ is equicontinuous for $t > 0$.

The following lemma is obvious.

Lemma 4 If the evolution system $\{U(t, s) : 0 \leq s \leq t < \infty\}$ is equicontinuous and $\eta \in L([0, b]; \mathbf{R}^+)$, then the set $\{\int_0^t U(t, s)x(s)ds : \|x(s)\| \leq \eta(s) \text{ for a.e., } 0 \leq s \leq t \leq b\}$ is equicontinuous.

3 Main results

Now we define the mild solution for the initial value problem (1.1)-(1.2).

Definition 6 A function $x : (-\infty, b] \rightarrow X$ is a mild solution of the initial value problem (1.1)-(1.2) if $x_0 = \varphi$, $x(\cdot)|_{[0, b]} \in C([0, b]; X)$ and

$$x(t) = U(t, 0)(\varphi(0) - h(0, \varphi)) + h(t, x_t) + \int_0^t U(t, s)A(s)h(s, x_s)ds + \int_0^t U(t, s)(f(s, x_s) + g(s, x_s))ds, \quad t \in J = [0, b].$$

For the system (1.1)-(1.2), we assume that the following hypotheses are satisfied:

(Hf) $f : J \times \mathcal{B} \rightarrow X$ satisfies the following condition:

(1) For each $x : (-\infty, b] \rightarrow X$, $x_0 \in \mathcal{B}$ and $x|_J \in C([0, b]; X)$, the function $f(\cdot, x)$ is strongly measurable for all $v \in \mathcal{B}$ and $f(t, \cdot)$ is continuous for a.e. $t \in [0, b]$;

(2) There exists an integrable function $\alpha : [0, b] \rightarrow [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega : [0, +\infty) \rightarrow (0, +\infty)$, such that

$$\|f(t, v)\| \leq \alpha(t)\Omega(\|v\|_{\mathcal{B}}), \quad \forall t \in [0, b], v \in \mathcal{B};$$

(3) There exists an integrable function $\eta : [0, b] \rightarrow [0, +\infty)$, such that

$$\chi(U(t, s)f(s, D)) \leq \eta(t) \sup_{-\infty \leq \theta \leq 0} \chi(D(\theta)), \quad \text{a.e. } s, t \in [0, b],$$

where $D(\theta) = \{v(\theta) : v \in D\}$.

(Hg) There exists an continuous function $L : [0, b] \rightarrow \mathbf{R}^+$, such that

$$\|g(t, u) - g(t, v)\| \leq L(t)\|u - v\|_{\mathcal{B}}, \quad \forall u, v \in \mathcal{B}.$$

(Hh) There exists a positive constant L_{h1} , L_{h2} and L_{h*} , such that

$$\begin{aligned} \|A(t)h(t, v)\| &\leq L_{h1}\|v\|_{\mathcal{B}} + L_{h2}, \quad (t, v) \in J \times \mathcal{B} \text{ and} \\ \|A(t_1)h(t_1, v_1) - A(t_2)h(t_2, v_2)\| &\leq L_{h*}(\|t_1 - t_2\| + \|v_1 - v_2\|_{\mathcal{B}}), \quad (t_i, v_i) \in J \times \mathcal{B}, \quad i = 1, 2. \end{aligned}$$

(H)(1) The evolution system $\{U(t, s) : 0 \leq s \leq t < \infty\}$ is equicontinuous and there exists a positive constant M , such that

$$(2) \frac{K_b M}{1 - \mu_2} \int_0^t \max \{ \alpha(t), L(t) \} ds < \int_c^{+\infty} \frac{ds}{\Omega(s) + s}, \text{ where}$$

$$\mu_1 = K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b L_{h2} (M_0 + M_b),$$

$$\mu_2 = K_b L_{h1} (M_0 + M_b), \quad c = \frac{\mu_1}{1 - \mu_2}, \quad K_b = \sup_{0 \leq t \leq b} K(t), \quad M_b = \sup_{0 \leq t \leq b} M(t).$$

$$(3) M \int_0^b (L_{h*} + L(s)) K(s) ds + \int_0^b \eta(s) ds < 1.$$

(4) $0 \in \rho(A(t))$ for $t \in J$, and there exist a constant $M_0 > 0$, such that $\|A^{-1}(t)\| \leq M_0$ for $t \in J$.

Let $y : (-\infty, b] \rightarrow X$ be the function defined by $y_0 = \varphi$ and $y(t) = U(t, 0)(\varphi(0) - h(0, \varphi))$ on J . Clearly, $\|y_t\|_{\mathcal{B}} \leq (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b M \|h(0, \varphi)\|$.

Now we are ready to establish our main results.

Theorem 5 *If the hypotheses (Hf) (Hg) (Hh) and (H) are satisfied, then there exists at least one mild solution to the initial value problem(1.1)-(1.2).*

Proof. Let $S(b)$ be the space $S(b) = \{x : (-\infty, b] \rightarrow X | x_0 = 0, x|_J \in C([0, b]; X)\}$ endowed with supremum norm $\|\cdot\|_b$. Let $\Gamma : S(b) \rightarrow S(b)$ be the map defined by

$$\Gamma x(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ h(t, x_t + y_t) + \int_0^t U(t, s) A(s) h(s, x_s + y_s) ds \\ + \int_0^t U(t, s) (f(s, x_s + y_s) + g(s, x_s + y_s)) ds, & t \in [0, b]. \end{cases} \quad (3.1)$$

Due to the fact that $\|x_t + y_t\|_{\mathcal{B}} \leq K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b \|x\|_t$, where $\|x\|_t = \sup_{0 \leq s \leq t} \|x(s)\|$, Γ is well defined and with values in $S(b)$. It is easy to see that if x is a fixed point of Γ , then $x + y$ is a mild solution to the initial value problem (1.1)-(1.2). In the sequel we will prove that there exists a fixed point of Γ by Darbo's fixed point theorem.

We first note that, Γ is continuous on the basis of the axioms of phase space, the Lebesgue dominated convergence theorem and the conditions (Hf), (Hg) and (Hh), we assert that the function $s \mapsto U(t, s) A(s) h(s, x_s + y_s)$ and $s \mapsto U(t, s) (f(s, x_s + y_s) + g(s, x_s + y_s))$ are integrable on $[0, t]$ for every $t > 0$ and every bounded $x \in S(b)$. In fact, let $\|x\|_b < k$, where k is a positive constant. In view of (Hh) we have

$$\begin{aligned} \|U(t, s) A(s) h(s, x_s + y_s)\| &\leq M(L_{h1} \|x_s + y_s\|_{\mathcal{B}} + L_{h2}) \\ &\leq M(L_{h1} (K_b M \|h(0, \varphi)\| \\ &\quad + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b k) + L_{h2}), \end{aligned}$$

$$\begin{aligned} \|U(t, s) (f(s, x_s + y_s) + g(s, x_s + y_s))\| \\ \leq M(\alpha(t) \Omega(\|x_s + y_s\|_{\mathcal{B}}) + L(t) \|x_s + y_s\|_{\mathcal{B}} + \|g(s, 0)\|) \\ \leq M(\alpha(t) \Omega(K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b k) \\ + L_g (K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b k) + \|g(s, 0)\|), \end{aligned}$$

where $L_g = \max_{t \in J} L(t)$. Hence the function $s \mapsto U(t, s) A(s) h(s, x_s + y_s)$ and $s \mapsto U(t, s) (f(s, x_s + y_s) + g(s, x_s + y_s))$ is integrable on $[0, t]$, for every $t > 0$.

Next, we show that the set $\{x \in C([0, b]; X) : x = \lambda \Gamma x\}$ is bounded for $0 < \lambda < 1$. Let x_λ be a solution of $x = \lambda \Gamma x$ for $0 < \lambda < 1$. Then

$$\|x_{\lambda t} + y_t\|_{\mathcal{B}} \leq K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b \|x_\lambda\|_t.$$

Let $v_\lambda(t) = K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} + K_b \|x_\lambda\|_t$ for each $t \in J$, then

$$\begin{aligned} \|x_\lambda(t)\| &= \|\lambda \Gamma x_\lambda(t)\| \leq \lambda \|\Gamma x_\lambda(t)\| \\ &\leq M \int_0^t (\alpha(s) \Omega(\|x_{\lambda s} + y_s\|_{\mathcal{B}}) + L(s) \|x_{\lambda s} + y_s\|_{\mathcal{B}}) ds \\ &\quad + L_{h1}(M_0 + Mb) v_\lambda(t) + L_{h2}(M_0 + Mb) + M \int_0^t \|g(s, 0)\| ds, \end{aligned}$$

$$\begin{aligned} \|x_\lambda\|_t &\leq M \int_0^t (\alpha(s) \Omega(\|x_{\lambda s} + y_s\|_{\mathcal{B}}) + L(s) \|x_{\lambda s} + y_s\|_{\mathcal{B}}) ds \\ &\quad + L_{h1}(M_0 + Mb) v_\lambda(t) + L_{h2}(M_0 + Mb) + M \int_0^t \|g(s, 0)\| ds, \end{aligned}$$

which implies that

$$\begin{aligned} v_\lambda(t) &\leq K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} \\ &\quad + K_b M \int_0^t (\alpha(s) \Omega(\|x_{\lambda s} + y_s\|_{\mathcal{B}}) + L(s) \|x_{\lambda s} + y_s\|_{\mathcal{B}}) ds \\ &\quad + L_{h1}(M_0 + Mb) v_\lambda(t) + L_{h2}(M_0 + Mb) + M \int_0^t \|g(s, 0)\| ds \\ &\leq K_b M \|h(0, \varphi)\| + (K_b M H + M_b) \|\varphi\|_{\mathcal{B}} \\ &\quad + K_b L_{h2}(M_0 + Mb) + K_b L_{h1}(M_0 + Mb) v_\lambda(t) \\ &\quad + K_b M \int_0^b \|g(s, 0)\| ds + K_b M \int_0^t (\alpha(s) \Omega(v_\lambda(s)) + L(s) v_\lambda(s)) ds. \end{aligned}$$

Consequently,

$$v_\lambda(t) \leq c + \frac{K_b M}{1 - \mu_2} \int_0^t (\alpha(s) \Omega(v_\lambda(s)) + L(s) v_\lambda(s)) ds. \quad (3.2)$$

Denoting by $\beta_\lambda(t)$ the right-hand side of (3.2), we get

$$\begin{aligned} \beta'_\lambda(t) &= \frac{K_b M}{1 - \mu_2} (\alpha(s) \Omega(v_\lambda(s)) + L(s) v_\lambda(s)) \\ &\leq \frac{K_b M}{1 - \mu_2} \max\{\alpha(t), L(t)\} (\Omega(\beta_\lambda(t)) + \beta_\lambda(t)), \end{aligned}$$

therefor,

$$\frac{\beta'_\lambda(t)}{\Omega(\beta_\lambda(t)) + \beta_\lambda(t)} \leq \frac{K_b M}{1 - \mu_2} \max\{\alpha(t), L(t)\}. \quad (3.3)$$

Integrating (3.3) and applying our hypothesis (H)(1), we obtain

$$\int_c^{\beta_\lambda(t)} \frac{ds}{\Omega(s) + s} \leq \frac{K_b M}{1 - \mu_2} \int_0^t \max\{\alpha(s), L(s)\} ds < \int_c^{+\infty} \frac{ds}{\Omega(s) + s},$$

which implies that $\beta_\lambda(t)$ is bounded on J . Thus, $v_\lambda(t)$ is bounded on J , and $x_\lambda(\cdot)$ is also bounded on J . Now, we show that Γ is χ -contraction. To do this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$ for $t \geq 0$, where

$$\begin{aligned} \Gamma_1 x(t) &= h(t, x_t + y_t) + \int_0^t U(t, s) A(s) h(s, x_s + y_s) ds + \int_0^t U(t, s) g(s, x_s + y_s) ds, \\ \Gamma_2 x(t) &= \int_0^t U(t, s) f(s, x_s + y_s) ds. \end{aligned}$$

First, we show that Γ_1 is lipschitzian on $S(b)$. Take $x_1, x_2 \in S(b)$ arbitrary, on account of definition 1 and hypotheses, we get that

$$\begin{aligned} \|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| &\leq \left\| \int_0^t U(t,s)A(s)(h(s, x_{1s} + y_s) - h(s, x_{2s} + y_s))ds \right\| \\ &\quad + \left\| \int_0^t U(t,s)(g(s, x_{1s} + y_s) - g(s, x_{2s} + y_s))ds \right\| \\ &\leq M \int_0^t L_{h^*}K(s) \sup_{0 \leq \tau \leq b} \|x_1(\tau) - x_2(\tau)\| ds \\ &\quad + M \int_0^t L(s)K(s) \sup_{0 \leq \tau \leq b} \|x_1(\tau) - x_2(\tau)\| ds \\ &\leq M \int_0^b (L_{h^*} + L(s))K(s)ds \|x_1 - x_2\|_b. \end{aligned}$$

Therefore

$$\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_b \leq M \int_0^b (L_{h^*} + L(s))K(s)ds \|x_1 - x_2\|_b. \tag{3.4}$$

for any $x_1, x_2 \in S(b)$, and Γ_1 is lipschitzian on $S(b)$ with Lipschitz constant $L' = M \int_0^b (L_{h^*} + L(s))K(s)ds$.

Next, take bounded subset $W \subset S(b)$ arbitrary. The hypothesis that $U(t, s)$ is equicontinuous implies the equicontinuity of the set $U(t, s)f(s, W_s + y_s)$. In view of Lemma 1,4 and (Hf)(3), we obtain that

$$\begin{aligned} \chi(\Gamma_2 w(t)) &= \chi\left(\int_0^t U(t,s)f(s, W_s + y_s)ds\right) \\ &\leq \int_0^t \eta(s) \sup_{-\infty < \theta \leq 0} \chi(W(s + \theta) + y(s + \theta))ds \\ &\leq \int_0^t \eta(s) \sup_{0 \leq \tau \leq s} \chi W(\tau)ds \leq \chi_C(W) \int_0^t \eta(s)ds, \end{aligned}$$

and hence

$$\chi_C(\Gamma_2 w) \leq \chi_C(W) \int_0^t \eta(s)ds \leq \chi_C(W) \int_0^b \eta(s)ds \tag{3.5}$$

for each bounded set $W \in C([0, b]; X)$.

Now, for each bounded set $W \in C([0, b]; X)$, we get from (3.4),(3.5) and Lemma 1 that

$$\begin{aligned} \chi_C(\Gamma W) &= \chi_C(\Gamma_1 W + \Gamma_2 W) \leq \chi_C(\Gamma_1 W) + \chi_C(\Gamma_2 W) \\ &\leq (L' + \int_0^t \eta(s)ds)\chi_C(W) \leq \chi_C(W). \end{aligned}$$

The hypothesis (H)(2) implies that Γ is χ - contraction. In view of Lemma 2 (Darbo fixed point Theorem), we conclude that Γ there exists at least one fixed point of Γ in $S(b)$. Let x be a fixed of Γ on $S(b)$, then $z = x + y$ is a mild solution of (1.1)-(1.2), which complete the proof. ■

4 An Example

Consider the model

$$\begin{cases} \frac{\partial}{\partial t} [v(t, \xi) - \int_{-\infty}^0 T(\theta)u(t, v(t + \theta, \xi))d\theta] \\ = a(t, \xi) \frac{\partial^2 v}{\partial \xi^2}(t, \xi) + \int_{-\infty}^0 P(\theta)r(t, v(t + \theta, \xi))d\theta \\ + \int_{-\infty}^0 Q(\theta)s(t, v(t + \theta, \xi))d\xi, t \in [0, b], \xi \in [0, \pi] \\ v(t, 0) = v(t, \pi) = 0, t \in [0, b], \\ v(\theta, \xi) = v_0(\theta, \xi), -\infty < \theta \leq 0, \xi \in [0, \pi], \end{cases} \tag{4.1}$$

where $a(t, \xi)$ is a continuous function and is uniformly Hölder continuous in t ; $T, P, Q : (-\infty, 0] \rightarrow \mathbf{R}$; $u, r, s : [0, +\infty) \times \mathbf{R} \rightarrow \mathbf{R}$ and $v_0 : (-\infty, 0] \times [0, \pi] \rightarrow \mathbf{R}$ are continuous functions.

Let $E = L^2([0, \pi], \mathbf{R})$ and define $A(t)$ by $A(t)w = a(t, \xi)w''$ with domain $D(A) = \{w \in E : w, w' \text{ are absolutely continuous, } w'' \in E, w(0) = w(\pi) = 0\}$. Then $A(t)$ generates an evolution system $U(t, s)$ satisfying assumptions (H1) and (H2) (see [14]).

For the phase space \mathcal{B} , we choose the well known space $BUC(\mathbf{R}^-, E)$: the space of uniformly bounded continuous functions endowed with the norm $\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)|$ for $\varphi \in \mathcal{B}$. If we put for $\varphi \in BUC(\mathbf{R}^-, E)$ and $\xi \in [0, \pi]$,

$$x(t)(\xi) = v(t, \xi), t \in [0, b], \xi \in [0, \pi],$$

$$\phi(\theta)(\xi) = v_0(\theta, \xi), -\infty < \theta \leq 0, \xi \in [0, \pi],$$

$$h(t, \varphi)(\xi) = \int_{-\infty}^0 T(\theta)u(t, \varphi(\theta)(\xi))d\theta, -\infty < \theta \leq 0, \xi \in [0, \pi],$$

$$f(t, \varphi)(\xi) = \int_{-\infty}^0 P(\theta)r(t, \varphi(\theta)(\xi))d\theta, -\infty < \theta \leq 0, \xi \in [0, \pi],$$

$$g(t, \varphi)(\xi) = \int_{-\infty}^0 Q(\theta)s(t, \varphi(\theta)(\xi))d\theta, -\infty < \theta \leq 0, \xi \in [0, \pi],$$

Then, (4.1) takes the abstract neutral perturbed evolution form (1.1)-(1.2). To show the existence of the mild solution to (4.1), we assume the following hypotheses:

(1) the functions u and s are Lipschitz with respect to its second argument, and constants $\text{lip}(u)$ and $\text{lip}(s)$ respectively.

(2) There exist $p \in L^1([0, +\infty), \mathbf{R}^+)$ and a nondecreasing continuous function $\psi : [0, +\infty) \rightarrow (0, \infty)$ such that $|r(t, u)| \leq p(t)\psi(|u|)$, for $t \in [0, +\infty)$, $u \in \mathbf{R}$.

(3) T, P and Q are integrable on $(-\infty, 0]$.

By the dominated convergence theorem, one can show that f is a continuous function from \mathcal{B} to E . Moreover the mapping h and g are Lipschitz continuous in its second argument, in fact, we have

$$|g(t, \varphi_1) - g(t, \varphi_2)| \leq \text{lip}(s) \int_{-\infty}^0 |Q(\theta)|d\theta |\varphi_1 - \varphi_2|, \varphi_1, \varphi_2 \in \mathcal{B},$$

$$|h(t, \varphi_1) - h(t, \varphi_2)| \leq M_0 L_* \text{lip}(u) \int_{-\infty}^0 |T(\theta)|d\theta |\varphi_1 - \varphi_2|, \varphi_1, \varphi_2 \in \mathcal{B}.$$

On the other hand, for $\varphi \in \mathcal{B}$ and $\xi \in [0, \pi]$ we have

$$|f(t, \varphi)(\xi)| \leq \int_{-\infty}^0 |p(t)P(\theta)|\psi(|\varphi(\theta)(\xi)|)d\theta.$$

Since the function is nondecreasing, it follows that

$$|f(t, \varphi)| \leq p(t) \int_{-\infty}^0 |P(\theta)|d\theta \psi(|\varphi|), \text{ for } \varphi \in \mathcal{B}.$$

Proposition 6 Assume the above hypotheses and the condition H in Theorem 3.2 hold, $\varphi \in \mathcal{B}$, then there exists at least one mild solution to system (4.1).

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