ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.3,pp.356-359

Some Solutions of 2-order Periodic Camassa-Holm Equation

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(Received 12 July 2008, accepted 21 November 2009)

Abstract: Higher-order Camassa-Holm equation is a generalization of Camassa-Holm equation. In this paper, for 2-order periodic Camassa-Holm equation, we construct some solutions.
Keywords: Camassa-Holm equation; periodic solution; Cauchy problem

1 Introduction

Degasperis and Procesi (see [1]) studied the following family of third order dispersive partial differential equation (PDE) conservation laws:

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x \tag{1}$$

where $\alpha, \gamma, c_1, c_2, c_3$ are real constants. They found that there are only there equations that satisfy the asymptotic integrability condition within this family: the Korteweg-de Vries equation, the Camassa-Holm equation, and the Degasperis-Procesi equation.

For $c_1 = -3c_3/2\alpha^2$ and $c_2 = c_3/2$ in Eq.(1), it became the Camassa-Holm equation modeling the unidirectional propagation of shallow water waves over a flat bottom, $u(t, x)$ standing for the fluid velocity at time in the spatial direction. The Camassa-Holm equation was also a model for the propagation of axially symmetric waves in hyperelastic rods. It had a bi-Hamiltionian structure and was completely integrable. The orbital stability of the peaked solitons was proved, and that of the smooth solitons.

The explicit interaction of the peaked solitons was given. The Cauchy problem of the Camassa-Holm equation had been studied extensively. It had been shown that the Camassa-Holm equation was locally well-posed with the initial data $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$. More interestingly, it had global strong solutions and also blow-up solutions in finite time with a different class of initial profiles in the Sobolev spaces $H^s(\mathbb{R}), s > \frac{3}{2}$. On the other hand, it had global weak solutions in $H^1(\mathbb{R})$. Some results related to Camassa-Holm equation can be found in [4],[5],[6],[7] and referees therein.

In [2], a family of higher-order Camassa-Holm equations are constructed as:

$$\partial_t u = B_k(u, u), \tag{2}$$

where $u = u(t, x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is the unknown function and

$$\begin{aligned} B_k(u, u) &:= A_k^{-1} C_k(u) - u\partial_x u, \\ A_k(u) &:= \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \\ C_k(u) &:= -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u). \end{aligned}$$

When $k = 0$ and $k = 1$, (2) becomes the inviscid Burgers equation [2]

$$\partial_t u + 3u\partial_x u = 0 \tag{3}$$

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Supported in part by the Science Foundation of Jiangsu University.(No. 07JDG035, No. o7JDG038).

IJNS.2009.12.15/289

and the Camassa-Holm equation [3]

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u \quad = \quad 2\partial_x u \partial_x^2 u + u\partial_x^3 u, \tag{4}$$

respectively. The Cauchy problem is studied in [1].

For general $k \geq 2$, the behavior of equation (1) is quite different from the behavior of Camassa-Holm equation (3). Equation (1) has solitons: $u(t,x) = ce^{-|x-ct|}, c \in \mathbb{R}$ and peakon solutions for periodic case. But it doesn't happen when $k \geq 2$, in this case the term $\partial_x(\partial_x^j u)^2$ for some $j \geq 2$ appears in the deformation of (1):

$$\partial_t A_k u = -uA_k(\partial_x u) - 2\partial_x u A_k(u). \tag{5}$$

Hence idiographic solutions are helpful and important for us to understand the behavior of equation (2).

In this paper, we construct some travelling wave solutions for periodic 2-order Camassa-Holm equation.

2 Travelling wave solution for higher-order Camassa-Holm equation

Let $\xi := \mu + \eta\sqrt{-1}(\mu, \eta \in \mathbb{R})$ be the $2(k+1)$-th root of unit 1, define real functions on $[0,\infty) \times \mathbb{R}$:

$$\begin{aligned} v_1 &= e^{\eta(x-ct)} \cos(\mu(x-ct)), \\ v_2 &= e^{-\eta(x-ct)} \cos(\mu(x-ct)), \\ v_3 &= e^{\eta(x-ct)} \sin(\mu(x-ct)), \\ v_4 &= e^{-\eta(x-ct)} \sin(\mu(x-ct)), \end{aligned}$$

then we have

$$A_k(v_1) = A_k(v_2) = A_k(v_3) = A_k(v_4) = 0,$$

all of them are solutions of (5)[also of (2)]. Where c is the speed of the travelling waves.

In fact, $v_1 = Re(e^{-\sqrt{-1}\xi(x-ct)})$, the real part of $e^{-\sqrt{-1}\xi(x-ct)}$, and

$$\begin{aligned} A_k(v_1) &= Re(A_k(e^{-\sqrt{-1}\xi(x-ct)})) \\ &= Re\left(e^{-\sqrt{-1}\xi(x-ct)} \sum_{j=0}^{k} (-1)^j(-\sqrt{-1}\xi)^{2j}\right) \\ &= Re(\left(e^{-\sqrt{-1}\xi(x-ct)} \sum_{j=0}^{k} \xi^{2j}\right) = 0. \end{aligned}$$

For v_2, v_3, v_4, the situation is similar.

3 Solutions for periodic 2-order Camassa-Holm equation

Define functions

$$w_1 = (v_1 + v_2)/2, w_2 = (v_1 - v_2)/2, w_3 = (v_3 + v_4)/2, w_4 = (v_3 - v_4)/2,$$

then, when $k = 2$, we have

$$w_1 = \cosh\left(\frac{\sqrt{3}}{2}(x-ct)\right)\cos\left(\frac{1}{2}(x-ct)\right) \qquad w_2 = -\sinh\left(\frac{\sqrt{3}}{2}(x-ct)\right)\cos\left(\frac{1}{2}(x-ct)\right)$$

$$w_3 = \cosh\left(\frac{\sqrt{3}}{2}(x-ct)\right)\sin\left(\frac{1}{2}(x-ct)\right) \qquad w_4 = -\sinh\left(\frac{\sqrt{3}}{2}(x-ct)\right)\sin\left(\frac{1}{2}(x-ct)\right)$$

Lemma 1 *There exist infinite many positive numbers x_0 such that*

$$\frac{\sinh(\frac{\sqrt{3}}{2}x_0)}{\cosh(\frac{\sqrt{3}}{2}x_0)} = \frac{\sin(x_0/2)}{\sqrt{3}\cos(x_0/2)} \tag{6}$$

Proof. By observation, $\frac{\sin(x)}{\sqrt{3}\cos(x)}$ runs between $-\infty$ and ∞ periodically, in another hand, $\frac{\sinh(x)}{\cosh(x)} > 0$ intends to 1 when x intends to ∞, hence the statement is true.

For any positive x_0 satisfying (7) and positive integer n, define

$$\begin{aligned} u &= u(t,x) : [0,\infty) \times \mathbb{R}/(x - ct + 2\mathbb{Z}x_0) \to \mathbb{R} \\ & (t,x) \longmapsto \gamma_{x_0} \cosh\left(\frac{\sqrt{3}}{2}(x-ct)\right) \cos\left(\frac{1}{2}(x-ct)\right) = \gamma_{x_0} w_1, \end{aligned}$$

for $x - ct \in [-x_0, x_0]$, where γ_{x_0} is to be determined.

Definition 1 *A distribution solution of (5) is a function*

$$u(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$$

satisfying (5) in the sense of distributions.

Proposition 2 *There exists γ_{x_0} such that u is a periodic distribution solution of (5).*

4 Proof for the result

By the definition, u and v are clearly continuous. Moreover, they are linear combinations of w_i's, hence they satisfy the equation (5) when $x - ct \in (-x_0, x_0)$ and $x - ct \in (-\pi, \pi)$, respectively.

Since

$$\begin{aligned} & \partial_x \left(\cosh\left(\frac{\sqrt{3}}{2}(x-ct)\right) \cos\left(\frac{1}{2}(x-ct)\right)\right) \\ = & \frac{\sqrt{3}}{2} \sinh\left(\frac{\sqrt{3}}{2}(x-ct)\right) \cos\left(\frac{1}{2}(x-ct)\right) - \frac{1}{2} \cosh\left(\frac{\sqrt{3}}{2}(x-ct)\right) \sin\left(\frac{1}{2}(x-ct)\right) \end{aligned}$$

which equals zero when $x - ct = \pm x_0$, so

$$\partial_x u = u_x.$$

More over,

$$\partial_x^2 u = u_{xx} = \frac{1}{2}u + \gamma_{x_0} \frac{\sqrt{3}}{2} w_4$$

as u_{xx} is even about $x - ct$, and

$$\partial_x^3 u = u_{xxx} = \frac{1}{2}u_x + \gamma_{x_0}(\frac{\sqrt{3}}{4}w_2 - \frac{3}{4}w_3)$$

which is not continuous, in the sense of distributions,

$$\partial_x^4 u - u_{xxxx} = \left(-2\gamma_{x_0} \cosh\frac{\sqrt{3}x_0}{2} \sin\frac{x_0}{2}\right) \delta_{x-ct-x_0}$$

Notice that u_{xxxx} is even about $x - ct$ hence continuous,

$$\partial_x^5 u - u_{xxxxx} = \left(-2\gamma_{x_0} \cosh\frac{\sqrt{3}x_0}{2} \sin\frac{x_0}{2}\right) \delta^{(1)}_{x-ct-x_0}$$

$$\partial_t \partial_x^4 u - u_{txxxx} = \left(2c\gamma_{x_0} \cosh \frac{\sqrt{3}x_0}{2} \sin \frac{x_0}{2}\right) \delta^{(1)}_{x-ct-x_0}$$

Consider distributions of equation (5), which holds if and only if

$$\begin{aligned} & \left(2c\gamma_{x_0} \cosh \frac{\sqrt{3}x_0}{2} \sin \frac{x_0}{2}\right) \delta^{(1)}_{x-ct-x_0} \\ = & -u\left(-2\gamma_{x_0} \cosh \frac{\sqrt{3}x_0}{2} \sin \frac{x_0}{2}\right) \delta^{(1)}_{x-ct-x_0} - 3u_x\left(-2\gamma_{x_0} \cosh \frac{\sqrt{3}x_0}{2} \sin \frac{x_0}{2}\right) \delta_{x-ct-x_0} \end{aligned}$$

holds at x_0, so

$$\left(2c\gamma_{x_0} \cosh \frac{\sqrt{3}x_0}{2} \sin \frac{x_0}{2}\right) \delta^{(1)}_{x-ct-x_0} = \left(2\gamma_{x_0}^2 \cosh^2 \frac{\sqrt{3}x_0}{2} \sin \frac{x_0}{2} \cos \frac{x_0}{2}\right) \delta^{(1)}_{x-ct-x_0}$$

it solves

$$\gamma_{x_0} = \frac{c}{\cosh \frac{\sqrt{3}x_0}{2} \cos \frac{x_0}{2}}.$$

Theorem 3 *For any $x_0 > 0$ satisfying (6),*

$$u = u(t,x) = \frac{c}{\cosh \frac{\sqrt{3}x_0}{2} \cos \frac{x_0}{2}} \cosh\left(\frac{\sqrt{3}}{2}(x-ct)\right) \cos\left(\frac{1}{2}(x-ct)\right)$$

is a periodic distribution solution of the 2-order Camassa-Holm equation.

Remark 4 *This construction can't be applied for $k \geq 3$.*

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