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A Functional Limit Theorem for Self-normalized Products of Sums under Dependence Assumption

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Abstract: Let $\{X, X_n, n \geq 1\}$ be a sequence of strictly stationary linear positive quadrant dependent, positive random variables in the domain of attraction of the normal law. Under some suitable conditions, the weak invariance principle for self-normalized products of partial sums is obtained.

Key words:invariance principle; self-normalized; linear positive quadrant dependence
MSC:60F15; 60F17

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of positive random variables and define the partial sum $S_n = \sum_{i=1}^n X_i$ and $V_n^2 = \sum_{i=1}^n (X_i - \overline{X})^2$ for $n \geq 1$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Arnold and Villaseñor [1] considered the limiting properties of sums of records and obtained the following version of the central limit theorem for i.i.d. exponential random variables with mean one,

$$\frac{\sum_{k=1}^n \log S_k - n\log n + n}{\sqrt{2n}} \xrightarrow{\mathcal{D}} \mathcal{N}, \quad as \quad n \to \infty.$$

Here and in the sequel, $\mathcal{N}$ denotes a standard normal random variable. Later Rempala and Wesoloski [7] removed the condition that the distribution of X_i is exponential and extended such a central limit theorem to general i.i.d positive random variable. Recently, Lixin Zhang and Wei Huang [8] extended this kind of results to the invariance principle. In this paper, we shall study the weak invariance principle for self-normalized products of sums under dependence assumption.

Two random variables X and Y are said to be positive quadrant dependent (PQD)if $P(X > x, Y > y) \geq P(X > x)P(Y > y)$ for all $x, y \in \mathcal{R}$. A sequence $\{X_n; n \geq 1\}$ is said to be linear positive quadrant dependent if for any disjoint finite subsets $A, B \subset \{1, 2, \ldots\}$ and any positive real numbers r_j,

$$\sum_{i\in A} r_i X_i \quad and \quad \sum_{j\in B} r_j X_j \quad are \quad PQD.$$

We also put

$$\begin{gathered} l(x) = E(X-\mu)^2 I(|X-\mu| \leq x), \\ b = inf\{x \geq 1; l(x) > 0\}, \\ A_n^2 = Var\Big(\sum_{i=1}^n (X_i-\mu) I(|X_i-\mu| \leq \eta_n)\Big), \end{gathered}$$

and

$$B_n^2 = \sum_{i=1}^n \mathsf{E}(X_i-\mu)^2 I(|X_1-\mu| \leq \eta_n),$$

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where $\mu = \mathsf{E}X$ and $\eta_i = \inf\left\{s : s \geq b+1, \frac{l(s)}{s^2} \leq \frac{1}{i}\right\}, i = 1, 2, \cdots$. It is easily seen that $\eta_n \leq \eta_{n+1}$ and $B_n^2 = nl(\eta_n) \sim \eta_n^2$ as $n \to \infty$. From now on, C denotes a constant whose value can differ from line to line. The followings are our main results.

Theorem 1 *Let $\{X_n; n \geq 1\}$ be a strictly stationary $LPQD$ sequence of positive random variables in the domain of attraction of the normal law with $E(X_1) = \mu > 0, E(X_1^2) < \infty$. Assume that*
(C_1) $\quad A_n^2 \sim \beta B_n^2 \quad for \quad 0 < \beta < \infty,$
(C_2) $\quad Cov(X_1, X_{n+1}) = O(n^{-1}(\log n)^{-2-\epsilon}) \quad for \quad \epsilon > 0$. *Then*

$$\left(\prod_{k=1}^{[nt]} \frac{S_k}{k\mu}\right)^{\frac{\mu}{V_n}} \xrightarrow{\mathcal{D}} \exp\left\{\beta \int_0^t \frac{W(x)}{x} dx\right\} \quad in\ D[0,1],\ as\ n \to \infty,$$

where $\{W(t), t \geq 0\}$ is a standard wiener process.

In particular, when we take $t = 1$, it yields the following remark 1.

Remark 2 *Under the assumptions of Theorem 1.1, we have that*

$$\left(\prod_{k=1}^{n} \frac{S_k}{k\mu}\right)^{\frac{\mu}{V_n}} \xrightarrow{\mathcal{D}} e^{\sqrt{2}\beta\mathcal{N}} \quad as\ n \to \infty,$$

where $\mathcal{N}$ is a standard normal random variable.

Remark 3 *Since $\int_0^1 \frac{W(x)}{x} dx$ is a normal random variable with*

$$\mathsf{E}(\int_0^1 \frac{W(x)}{x} dx) = 0,$$

and

$$\mathsf{E}(\int_0^1 \frac{W(x)}{x} dx)^2 = 2,$$

Remark 1 is trivial.

The following example comes from Yunxia Li and Jianfeng Wang [6].

Remark 4 *A finite family of random variables $\{X_i; 1 \leq i \leq n\}$ is said to be positively associated (PA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots\}$,*

$$Cov\{f(X_i; i \in A), g(X_j; j \in B)\} \geq 0,$$

whenever f and g are coordinatewise increasing and the covariance exists. A PA sequence is obviously a LPQD sequence, the following example shows that LPQD does not imply PA: Consider three discrete random variables with joint density $p(x, y, z) := P(X = x, Y = y, Z = z)$.$p(2, 2, 1) = p(3, 2, 1) = p(2, 3, 1) = p(3, 3, 1) = p(1, 1, 2) = p(2, 1, 2) = p(3, 1, 2) = p(1, 2, 2) = p(1, 3, 2) = \frac{1}{17}$ and$= p(1, 1, 1) = p(3, 3, 2) = \frac{4}{17}$. A lengthy verification shows that $\{X, Y, Z\}$ is LPQD. But, $\{X, Y, Z\}$ is not PA since $P(X > 1, Y > 1, Z > 1) = \frac{4}{17} < P(X > 1, Y > 1)P(Z > 1) = \frac{72}{289}$.

2 Proof of Theorem 1

We state some lemmas before showing the proof of Theorem 1. The following lemma comes from Csörgő, Szyszkowicz and Qiying Wang [4].

Lemma 5 *If $E(X) = 0$, then the following statements are equivalent:*
(1) X is in the domain of attraction of the normal law;
(2) $xE|X|I(|X| > x) = o(l(x))$;
(3) $E|X|^\alpha I(|X| \leq x) = o(x^{\alpha-2}l(x)) \ \ for \ \ \alpha > 2$.

Lemma 6 *Let $\{X_n; n \geq 1\}$ be a strictly stationary LPQD sequence of positive random variables with $E(X_1) = \mu > 0, E(X_1^2) < \infty$. Assume that condition (C_2) holds, then*

$$\frac{S_n}{n} \to 0 \quad a.s. \quad as \quad n \to \infty.$$

Proof. see Theorem 1 in Birkel [3]. ∎

Lemma 7 *Under the assumptions of Theorem 1 We have that*

$$\frac{\mu}{V_n} \sum_{k=1}^{n} (\frac{S_k}{\mu k} - 1)^2 \xrightarrow{P} 0. \tag{1}$$

Proof. Put $X_i^*(n) = (X_i - \mu)I(|X_i - \mu| \leq \eta_n)$ and $S_k^*(n) = \sum\limits_{i=1}^{k} X_i^*(n)$. First, we show $V_n^2/B_n^2 \xrightarrow{P} 1$. To obtain this, we need the following fact,

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sum_{i=1}^{n}(X_i - \mu)^2} \to 1 \;\; a.s.(n \to \infty). \tag{2}$$

Indeed

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sum_{i=1}^{n}(X_i - \mu)^2} = \frac{\sum_{i=1}^{n}(X_i - \mu)^2 - n(\mu - \overline{X})^2}{\sum_{i=1}^{n}(X_i - \mu)^2} = 1 - \frac{n(\mu - \overline{X})^2}{\sum_{i=1}^{n}(X_i - \mu)^2}, \tag{3}$$

we choose two constants $M > 0$ and $0 < \delta < 1$ such that $\mathsf{P}(|X - \mu| > M) > \delta > 0$, and hence it follows from the strong law of large numbers that for n large enough,

$$\begin{aligned}
\frac{n(\mu - \overline{X})^2}{\sum_{i=1}^{n}(X_i - \mu)^2} &\leq \frac{(\mu - \overline{X})^2}{n^{-1}\sum_{i=1}^{n}(X_i - \mu)^2 I(|X_i - \mu| > M)} \\
&\leq \frac{(\mu - \overline{X})^2}{M^2 n^{-1}\sum_{i=1}^{n} I(|X_i - \mu| > M)} \\
&= \frac{o(1)}{M^2\big(\mathsf{P}(|X - \mu| > M) + o(1)\big)} \\
&= o(1) \quad a.s.,
\end{aligned} \tag{4}$$

which together with (3) implies that as $n \to \infty$,

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sum_{i=1}^{n}(X_i - \mu)^2} = \frac{V_n^2}{\sum_{i=1}^{n}(X_i - \mu)^2} \to 1 \;\; a.s. \tag{5}$$

and

$$\begin{aligned}
\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{B_n^2} - 1 &= \frac{\sum\limits_{i=1}^{n}\Big((X_i - \mu)^2 I(|X_i - \mu| \leq \eta_n) - \mathsf{E}(X_i - \mu)^2 I(|X_i - \mu| \leq \eta_n)\Big)}{B_n^2} \\
&\quad + \frac{\sum_{i=1}^{n}(X_i - \mu)^2 I(|X_i - \mu| > \eta_n)}{B_n^2} \\
&:= P_1 + P_2.
\end{aligned}$$

For P_1, from the condition $Cov(X_1, X_{n+1}) = O(n^{-1}(\log n)^{-2-\epsilon})$ for $\epsilon > 0$ and by lemma 5, it follows that

$$\mathsf{P}(|P_1| \geq \varepsilon) \leq C\frac{n\mathsf{E}(X_1 - \mu)^4 I(|X_1 - \mu| \leq \eta_n)}{B_n^4} = \frac{n}{B_n^4} \cdot \eta_n^2 l(\eta_n) \cdot \frac{\mathsf{E}(X_1 - \mu)^4 I(|X_1 - \mu| \leq \eta_n)}{\eta_n^2 l(\eta_n)} = o(1).$$

For P_2, by Lemma 5, we have that

$$\begin{aligned}
&P(\frac{\sum_{i=1}^{n}(X_i-\mu)^2 I(|X_i-\mu|>\eta_n)}{B_n^2}\geq\varepsilon)\\
&\leq P((B_n^{-1}\sum_{i=1}^{n}|X_i-\mu|I(|X_i-\mu|>\eta_n))^2\geq\varepsilon)\\
&\leq C\frac{nE|X_1-\mu|I(|X_1-\mu|>\eta_n)}{B_n}\\
&=\frac{nl(\eta_n)}{B_n\eta_n}\cdot\frac{\eta_n E(|X_1-\mu|)I(|X_1-\mu|>\eta_n)}{l(\eta_n)}\\
&=o(1).
\end{aligned}$$

Thus, these imply

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{B_n^2}\xrightarrow{P}1,$$

which coupled with (5) leads to

$$\frac{V_n^2}{B_n^2}\xrightarrow{P}1. \tag{6}$$

To complete the proof of the Lemma 7, we only need to prove

$$\frac{\mu}{B_n}\sum_{k=1}^{n}(\frac{S_k}{\mu k}-1)^2\xrightarrow{P}0. \tag{7}$$

Indeed, denote $C_k=\frac{S_k}{k\mu}, k=1,2,\cdots$. By the strong law of large numbers for $LPQD$ sequence, it follows that for any $\delta>0$, there exists a positive integer R such that

$$\mathsf{P}(\sup_{k\geq R}|C_k-1|>\delta)<\delta.$$

Hence, there exist two sequences $\{\delta_m\}\searrow 0$ ($\delta_1=1/2$) and $\{R_m^*\}\nearrow\infty$ such that

$$\mathsf{P}(\sup_{k\geq R_m^*}|C_k-1|>\delta_m)<\delta_m.$$

The strong law of large numbers also implies that there exists a sequence $\{R_m'\}\nearrow\infty$ such that

$$\sup_{k\geq R_m'}|C_k-1|\leq 1/m \quad a.s.$$

Thus in the sequel, we take $R_m=\max(R_m^*,R_m')$, and it leads to

$$\mathsf{P}(\sup_{k\geq R_m}|C_k-1|>\delta_m)<\delta_m \text{ and } \sup_{k\geq R_m}|C_k-1|\leq 1/m \ a.s.$$

So, by the fact $V_n^2 \xrightarrow{P} \infty$, we have that

$$
\begin{aligned}
&P(\frac{\mu}{B_n}\sum_{k=1}^{n}(\frac{S_k}{\mu k}-1)^2 \geq \varepsilon)\\
&\leq P(\frac{\mu}{B_n}\sum_{k=R_m+1}^{n}(\frac{S_k}{\mu k}-1)^2 \geq \varepsilon, \sup_{k\geq R_m}|C_k-1|\leq \delta_m)+P(\sup_{k\geq R_m}|C_k-1|>\delta_m)\\
&\leq P(\frac{\mu}{B_n}\sum_{k=R_m+1}^{n}(\frac{S_k}{\mu k}-1)^2 I(\sup_{k\geq R_m}|C_k-1|\leq \delta_m)\geq \varepsilon)+\delta_m\\
&\leq \mathsf{P}(\frac{C\delta_m}{\sqrt{B_n^2}}\sum_{k=R_m+1}^{n}\frac{1}{k}\big|S_k^*(n)-\mathsf{E}S_k^*(n)\big|\geq \varepsilon/2)+\mathsf{P}\bigg(\frac{C\delta_m}{\sqrt{B_n^2}}\sum_{k=R_m+1}^{n}\frac{1}{k}\bigg|\sum_{i=1}^{k}\bigg((X_j-\mu)I(|X_j-\mu|>\eta_n)\\
&\qquad -\mathsf{E}(X_j-\mu)I(|X_j-\mu|>\eta_n)\bigg)\bigg|\geq \varepsilon/2\bigg)+\delta_m\\
&:= P_3+P_4+\delta_m.
\end{aligned}
$$

For P_3, by Cauchy-Schwarz inequality, and the condition (C_1), it follows that

$$
P_3 \leq \frac{C\delta_m}{\sqrt{nl(\eta_n)}}\sum_{k=1}^{n}\frac{1}{k}\sqrt{Var(S_k^*(n))}\leq \frac{C\delta_m}{\sqrt{nl(\eta_n)}}\sum_{k=1}^{n}\sqrt{\frac{l(\eta_n)}{k}}\leq C\delta_m,
$$

which implies $P_3 \to 0$ by letting $n\to\infty$ and then $m\to\infty$.

For P_4, by $Markov's$ inequality and Lemma 5, we have that

$$
P_4 \leq \frac{CnE|X_1-\mu|I(|X_1-\mu|>\eta_n)}{B_n}=\frac{Cnl(\eta_n)}{B_n\eta_n}\cdot\frac{\eta_n E|X_1-\mu|I(|X_1-\mu|>\eta_n)}{l(\eta_n)}\to 0,
$$

as $n\to\infty$. The proof is completed. □ ■

Proof of Theorem 1 Denote $C_k=\frac{S_k}{k\mu}, k=1,2,\cdots$. By the strong law of large numbers for $LPQD$ sequence, and we take $R_m=\max(R_m^*, R_m')$, it follows that

$$
\mathsf{P}(\sup_{k\geq R_m}|C_k-1|>\delta_m)<\delta_m \text{ and } \sup_{k\geq R_m}|C_k-1|\leq 1/m \ \ a.s.
$$

The detail is omitted for sake of avoiding the repetitions. For any real x, write

$$
\begin{aligned}
P(\frac{\mu}{\beta V_n}\sum_{k=1}^{[nt]}\log(C_k)\leq x) =&P(\frac{\mu}{\beta V_n}\sum_{k=1}^{[nt]}\log(C_k)\leq x, \sup_{k\geq R_m}|C_k-1|>\delta_m)\\
&+P(\frac{\mu}{\beta V_n}\sum_{k=1}^{[nt]}\log(C_k)\leq x, \sup_{k\geq R_m}|C_k-1|\leq \delta_m)\\
=&A_{m,n}+B_{m,n}
\end{aligned}
$$

and $A_{m,n}<\delta_m$.

To compute $B_{m,n}$, we will use the expansion of the logarithm: $\log(1+x) = x - \frac{x^2}{2(1+\theta x)^2}$, where $\theta \in (0,1)$ depends on $|x| < 1$. Thus

$$\begin{aligned}
B_{m,n} =& P(\frac{\mu}{\beta V_n}\sum_{k=1}^{[nt]}\log(C_k) \le x, \sup_{k\ge R_m}|C_k-1| \le \delta_m) \\
=& P(\frac{\mu}{\beta V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\log(1+C_k-1) \le x, \sup_{k\ge R_m}|C_k-1|\le\delta_m) \\
=& P(\frac{\mu}{\beta V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}(C_k-1) \\
& - \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2} \le x, \sup_{k\ge R_m}|C_k-1|\le\delta_m) \\
=& P(\frac{\mu}{\beta V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}(C_k-1) \\
& - \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2}I(\sup_{k\ge R_m}|C_k-1|\le\delta_m) \le x) \\
& - P(\frac{\mu}{\beta V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}\log(C_k) + \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}(C_k-1) \le x, \sup_{k\ge R_m}|C_k-1| > \delta_m) \\
=& D_{m,n} + E_{m,n},
\end{aligned}$$

where $\theta_k (k = 1, \ldots, [nt])$ are (0-1)-valued and $E_{m,n} < \delta_m$.

Now, we rewrite $D_{m,n}$ as

$$\begin{aligned}
D_{m,n} =& P(\frac{\mu}{\beta V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}(\log(C_k) - C_k + 1) + \frac{\mu}{\beta V_n}\sum_{k=1}^{[nt]}(C_k-1) \\
& - \frac{\mu}{\beta V_n}\sum_{k=(R_m\wedge([nt]-1))+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2}I(\sup_{k\ge R_m}|C_k-1|\le\delta_m) \le x).
\end{aligned}$$

Observe that, for any fixed m, it is easy to obtain

$$\frac{\mu}{\beta V_n}\sum_{k=1}^{R_m\wedge([nt]-1)}(\log(C_k) - C_k + 1) \xrightarrow{P} 0 \quad \text{as } n \to \infty, \tag{8}$$

as $n \longrightarrow \infty$ by noting that $V_n^2 \xrightarrow{P} \infty$, where $V_n^2 = \sum_{k=1}^{n}(X_i - \bar{X})^2$.

If $R_m \ge [nt] - 1$, then

$$\frac{\mu}{\beta V_n}\frac{(C_{[nt]}-1)^2}{2(1+(C_{[nt]}-1)\theta_{[nt]})^2} \overset{a.s.}{\le} \frac{C}{V_n} \xrightarrow{P} 0$$

as $n \to \infty$.

If $R_m < [nt] - 1$, then $R_m + 1 < [nt]$. Denote

$$F_{m,n} := \frac{\mu}{\beta V_n}\sum_{k=R_m+1}^{[nt]}\frac{(C_k-1)^2}{2(1+\theta_k(C_k-1))^2}I(\sup_{k\ge R_m}|C_k-1|\le\delta_m),$$

by observing that $\frac{x^2}{(1+\theta x)^2} \le 4x^2$, and by Lemma 4, then imply that $F_{m,n} \xrightarrow{P} 0$.

Finally, to finish the proof, it is sufficient to show that

$$Y_n(t) := \frac{\mu}{\beta V_n} \sum_{k=1}^{[nt]} (C_k - 1) = \frac{1}{\beta V_n} \sum_{k=1}^{[nt]} \left(\frac{S_k - k\mu}{k} \right) \xrightarrow{\mathcal{D}} \int_0^t \frac{W(x)}{x} dx. \tag{9}$$

Denote

$$H_\epsilon(f)(t) = \begin{cases} \int_\epsilon^t \frac{f(x)}{x} dx, & t > \epsilon; \\ 0, & 0 \le t \le \epsilon, \end{cases}$$

and

$$Y_{n,\epsilon}(t) = \begin{cases} \frac{\mu}{\beta V_n} \sum\limits_{k=[n\epsilon]+1}^{[nt]} (C_k - 1), & t > \epsilon; \\ 0, & 0 \le t \le \epsilon. \end{cases}$$

It is readily seen that

$$\max_{0 \le t \le 1} \left| \int_0^t \frac{W(x)}{x} dx - H_\epsilon(W)(t) \right| = \sup_{0 \le t \le \epsilon} \left| \int_0^t \frac{W(x)}{x} dx \right| \to 0 \quad a.s. \quad \text{as } \epsilon \to 0. \tag{10}$$

Note that

$$\max_{0 \le t \le \epsilon} |Y_n(t) - Y_{n,\epsilon}(t)| = \max_{0 \le t \le \epsilon} \frac{1}{\beta V_n} \sum_{k=1}^{[nt]} \frac{|S_k - k\mu|}{k} \le \frac{1}{\beta V_n} \sum_{k=1}^{[n\epsilon]} \frac{|S_k - k\mu|}{k},$$

then, for any $\epsilon_1 > 0$, by the Cauchy-Schwarz inequality and (6), it follows that

$$\begin{aligned} \lim_{\epsilon \to 0} \limsup_{n \to \infty} P(\max_{0 \le t \le \epsilon} |Y_n(t) - Y_{n,\epsilon}(t)| \ge \varepsilon_1) &\le \lim_{\epsilon \to 0} \limsup_{n \to \infty} P(\frac{1}{V_n} \sum_{k=1}^{[n\epsilon]} \frac{|S_k - \mu k|}{k} \ge \varepsilon_1) \\ &\le \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{C}{B_n} \sum_{k=1}^{[n\epsilon]} \frac{E|S_k - \mu k|}{k} \\ &\le \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{C}{B_n} \sum_{k=1}^{[n\epsilon]} \frac{1}{\sqrt{k}} (\mathsf{Var}[\frac{S_k - \mu k}{\sqrt{k}}])^{1/2} \\ &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{C}{B_n} \sum_{k=1}^{[n\epsilon]} \frac{1}{\sqrt{k}} \\ &\le \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{C}{B_n} \sqrt{[n\epsilon]}. \end{aligned} \tag{11}$$

where, we need the following fact for $LPQD$ sequence. In fact, by condition (C_2) and the stationarity we get

$$Var(S_k) = kVar(X_1) + \sum_{j=2}^{k} (k+1-j) Cov(X_1, X_j) \le Ck.$$

Furthermore, we can obtain

$$\begin{aligned} &\sup_{\epsilon \le t \le 1} \frac{1}{\beta V_n} \left| \sum_{k=[n\epsilon]+1}^{[nt]} \frac{S_k - \mu k}{k} - \int_{n\epsilon}^{nt} \frac{S_{[x]} - [x]\mu}{x} dx \right| \\ &\le \sup_{\epsilon \le t \le 1} \frac{1}{\beta V_n} \left| \int_{[n\epsilon]+1}^{[nt]+1} \frac{S_{[x]} - [x]\mu}{[x]} dx - \int_{n\epsilon}^{nt} \frac{S_{[x]} - [x]\mu}{x} dx \right| \\ &\le \frac{1}{\beta V_n} \left| \int_{n\epsilon}^{[n\epsilon]+1} \frac{S_{[x]} - [x]\mu}{x} dx \right| + \sup_{\epsilon \le t \le 1} \frac{1}{\beta V_n} \left| \int_{nt}^{[nt]+1} \frac{S_{[x]} - [x]\mu}{x} dx \right| \\ &\quad + \sup_{\epsilon \le t \le 1} \frac{1}{\beta V_n} \left| \int_{[n\epsilon]+1}^{[nt]+1} (S_{[x]} - [x]\mu)(\frac{1}{x} - \frac{1}{[x]}) dx \right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\max_{k\le n}|S_k-\mu k|}{\beta V_n}\sup_{\epsilon\le t\le 1}\left(\frac{2}{n\epsilon}+\frac{2}{nt}+\frac{1}{n\epsilon}\right)\\
&\le C\frac{\max_{k\le n}|S_k-\mu k|}{nV_n}\\
&\le C\frac{\max_{k\le n}\sum_{i=1}^{k}|X_i-\mu|}{nV_n}=\frac{C}{V_n}\frac{\sum_{i=1}^{n}|X_i-\mu|}{n}\xrightarrow{a.s.}0.
\end{aligned} \tag{12}
$$

Therefore, uniformly for $t\in[\epsilon,1]$, we have

$$
\frac{1}{\beta V_n}\sum_{k=[n\epsilon]+1}^{[nt]}\frac{S_k-\mu k}{k}=\frac{1}{\beta V_n}\int_{n\epsilon}^{nt}\frac{S_{[x]}-[x]\mu}{x}dx+o_P(1)=\int_{\epsilon}^{t}\frac{W_n(t)}{x}dx+o_P(1), \tag{13}
$$

where $W_n(t):=\frac{S_{[nt]}-[nt]\mu}{V_n}$. Notice that $H_\epsilon(.)$ is a continuous mapping on the space $D[0,1]$. Thus using the continuous mapping theorem(c.f., Theorem 2.7 of Billingsley(1999)) it follows that

$$
Y_{n,\epsilon}(t)=H_\epsilon(W_n)(t)+o_P(1)\xrightarrow{\mathcal{D}}H_\epsilon(W)(t)\quad \text{in D[0,1], as } n\to\infty. \tag{14}
$$

Hence, combining (10), (11) and (14), coupled with Theorem 3.2 of Billingsley [2], (9) holds true. The proof is completed.

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References

[1] B. C. Arnold, J. A. Villaseñor: The asymptotic distritution of sums of records. *Extremes.* 1: 351-363(1998)

[2] P. Billingsley: Convergence of probability measures. *Wiley: New York*(1999)

[3] T. Birkel: A note on the strong law of large numbers for positively dependent random variable. *Statist. Probab. Lett.* 7: 17-20(1989)

[4] M. Csörgő, B. Szyszkowicz, Qiying. Wang: Donsker's theorem for self-normalized partial sums processes. *Ann. Probab.* 31: 1228-1240(2003)

[5] P. S. Griffin, J. D. Kuelbs: Self-normalized laws of the iterater logarithm. Ann. Probab. 17:1571-1601(1989)

[6] Yunxia. Li, Jianfeng. Wang: The law of the iterated logarithm for positively dependent random variables. *J. Math. Anal. Appl.* 339: 259-265(2008)

[7] G. Rempala, J. Wesolowski: Asymptotics for products of sums and U-statistics. *Elect. Comm. in. Probab.* 7: 47-54(2002)

[8] Lixin. Zhang, Wei. Huang: A note on the invariance principle of the product of sums of random variables. *Elect. Comm. in. Probab.* 12: 59-64(2007)