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Vibration Control for the Primary Resonance of the Duffing Oscillator by a Time Delay State Feedback

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Abstract: The primary resonance of an externally excited Duffing oscillator under feedback control with time delay is investigated. By means of the asymptotic perturbation method, two slow-flow equations on the amplitude and phase of the oscillator are obtained and the external excitation-response curves are shown. The stable condition of constant solution and the Hopf bifurcation condition to Duffing system are found, and it confirms that appropriate choice for the feedback gains and time delay can exclude the possibility of modulated motion and reduce the amplitude peak of the primary resonance.

Keywords:duffing oscillator; time delay; asymptotic perturbation method; stability analysis

1 Introduction

The control of resonantly forced systems has been investigated in various engineering fields in the last decades. It has become an urgent problem to consider the time-delay in the controllers and the actuators with increasing strict requirements for control speed and system performance. For example, digital controllers and reconstruction filters exhibit a certain time delay during operation. The controlled mechanic systems with time delay have been investigated by many researchers. For instance, a nonlinear active vibration absorber coupled with the plant through user-defined cubic nonlinearity was obtained in [1], a nonlinear delayed dynamic system of one dimension which may exhibit chaotic behavior was studied in [2,3], and the stability analysis and control of nonlinear system was investigated in [4-6]. Moiola, Chiacchiarini and Desagest [7] established the Hopf bifurcations resulting from nonlinear feedback systems with time delay, and they demonstrated that appropriate choices of the feedback gains and the time delay are possible for a better vibration control by using the multiple scales method [8,9]. In this paper, we consider the effect of time delay in a Duffing oscillator under an external excitation. The dynamics of an externally excited Duffing oscillator under time delay control is described by

$$\ddot{u}(t) + 2\zeta\omega_0\dot{u}(t) + \omega_0^2 u(t) + \varepsilon\omega_0^2 u^3(t) - F\cos\omega t + Bu(t-T) + C\dot{u}(t-T) = 0, \tag{1}$$

where dot denotes differentiation with respect to time and parameter ω_0 is the natural frequency and the external excitation frequency is $\omega \approx \omega_0$ so that the stochastic resonance can be enhanced. The paper is organized as follows. In section 2, a lowest order approximate solution of the nonlinear oscillator (1) is constructed by using the asymptotic perturbation method [10]. In section 3, the stability analysis and parametric resonance control are carried out, and the conclusion and further discussion are given in the section 4.

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2 The APD and lowest order approximate solution

In this section we discuss the case of primary resonance and let $\omega = \omega_0 + \varepsilon\sigma$, where $\varepsilon = O(1)$ and σ is the detuning parameter. We consider the case of small damping, weak non-linearity, weak feedback and soft excitation, i.e.

$$\zeta\omega_0 = \varepsilon\mu,\ \mu = O(1),\ F = \varepsilon f,\ \omega = \omega_0 + \varepsilon\sigma,\ f = O(1),\ \sigma = O(1).$$

Then Equation (1) is modified by

$$\ddot{u}(t) + 2\varepsilon\mu\dot{u}(t) + \omega_0^2 u(t) + \varepsilon\omega_0^2 u^3(t) - \varepsilon f\cos(\omega_0 t + \varepsilon\sigma t) + \varepsilon B u(t-T) + \varepsilon C\dot{u}(t-T) = 0. \tag{2}$$

The slow temporal scale $\tau = \varepsilon t$, which is associated with modulations in the amplitude and the phase of the solution, is used to describe the parametric resonance and modifications induced by non-linearity. The approximate solution $u(t)$ is given by

$$u(t) = \sum_{n[odd]=-\infty}^{+\infty} \varepsilon^{\gamma_n}\Psi_n(\tau,\varepsilon)\exp(in\omega t),$$

where $\gamma_n = |n| - 1$ and $\Psi_n(\tau,\varepsilon) = \Psi^*_{-n}(\tau,\varepsilon)$ at a real $u(t)$. The function $\Psi_n(\tau,\varepsilon)$ depends on the parameter ε and we suppose that the limit as $\varepsilon \to 0$ exists and is finite. The solution is then a Fourier expansion in which the coefficients vary slowly in time. The lowest order terms correspond to the harmonic solution of the linear problem. Evolution equations for the amplitudes of the harmonic terms are then derived by substituting the expression of the solution into the original equations and projecting onto each Fourier mode.

Considering the case of $n = 1$, $u(t)$ can be described by

$$u(t) = \Psi(\tau,\varepsilon)\exp(i\omega_0 t). \tag{3}$$

Substituting (3) into (2), considering the coefficients of the most important Fourier mode $n = 1$ and collecting terms of the same power of ε yields, to order ε,

$$2i\omega_0\frac{d\Psi}{d\tau} + 2\mu i\omega_0\Psi + 3\omega_0^2|\Psi|^2\Psi - \frac{1}{2}f\exp(i\sigma\tau) + B\Psi\exp(-i\omega_0 T) + Ci\omega_0\Psi\exp(-i\omega_0 T) = 0. \tag{4}$$

To analyze the combined effects of the no-linearity, the primary resonance and the delay control, we substitute the polar form

$$\Psi = \rho\exp(i\theta). \tag{5}$$

Substituting (5) into (4), separate real and imaginary parts and obtain

$$\begin{cases} \dfrac{d\rho}{d\tau} + \rho\mu - \dfrac{B\rho}{2\omega_0}\sin\omega_0 T + \dfrac{C\rho}{2}\cos\omega_0 T - \dfrac{f}{4\omega_0}\sin(\sigma\tau - \theta) = 0, \\ \rho\dfrac{d\theta}{d\tau} - \dfrac{3}{2}\omega_0\rho^3 - \dfrac{B\rho}{2\omega_0}\cos\omega_0 T - \dfrac{C\rho}{2}\sin\omega_0 T + \dfrac{f}{4\omega_0}\cos(\sigma\tau - \theta) = 0. \end{cases} \tag{6}$$

To simplify (6), we define

$$K = \sqrt{(\frac{C}{2})^2 + (\frac{B}{2\omega_0})^2},\ \cos\varphi = \frac{B}{2\omega_0 K},\ \sin\varphi = \frac{C}{2K}. \tag{7}$$

Substituting (7) into (6), we have

$$\begin{cases} \dfrac{d\rho}{d\tau} + \rho\mu - \rho K\sin(\omega_0 T - \varphi) - \dfrac{f}{4\omega_0}\sin(\sigma\tau - \theta) = 0, \\ \rho\dfrac{d\theta}{d\tau} - \dfrac{3}{2}\omega_0\rho^3 - \rho K\cos(\omega_0 T - \varphi) + \dfrac{f}{4\omega_0}\cos(\sigma\tau - \theta) = 0. \end{cases} \tag{8}$$

Let $\Omega = \sigma\tau - \theta$. Then it follows from (8) that

$$\begin{cases} \dfrac{d\rho}{d\tau} + \rho\mu - \rho K \sin(\omega_0 T - \varphi) - \dfrac{f}{4\omega_0}\sin\Omega = 0, \\ \rho\dfrac{d\Omega}{d\tau} - \rho\sigma + \dfrac{3}{2}\omega_0\rho^3 + \rho K\cos(\omega_0 T - \varphi) - \dfrac{f}{4\omega_0}\cos\Omega = 0. \end{cases} \tag{9}$$

Equations (9) represents a system defined with first order ODE, governing the amplitude and phase of the approximate solution expressed by

$$u(t) = \rho(\varepsilon t)\cos(\omega t - \Omega(\varepsilon t)).$$

3 Stability analysis and parametric resonance control

Letting $\dfrac{d\rho}{d\tau} = \dfrac{d\theta}{d\tau} = 0$, we find the external excitation response curve for the steady state solution amplitude corresponding to a periodic response of the starting system

$$f = 4\omega_0\rho[(\mu - K\sin(\omega_0 T - \varphi))^2 + (\sigma - \frac{3}{2}\omega_0\rho^2 - K\cos(\omega_0 T - \varphi))^2]^{0.5}. \tag{10}$$

So we have

$$F = 4\omega_0\rho[(\zeta\omega_0 - \varepsilon K\sin(\omega_0 T - \varphi))^2 + ((\omega - \omega_0) - \frac{3}{2}\varepsilon\omega_0\rho^2 - \varepsilon K\cos(\omega_0 T - \varphi))^2]^{0.5}. \tag{11}$$

Setting $\dfrac{d\rho}{d\tau} = \dfrac{d\theta}{d\tau} = 0$ in (9), we find that the constant solution (ρ, Ω) satisfy the following equations:

$$\begin{cases} \rho\mu + \rho K\sin\varphi - \dfrac{f}{4\omega_0}\sin\Omega = 0, \\ -\sigma + \dfrac{3}{2}\omega_0\rho^2 + \rho K\cos\varphi - \dfrac{f}{4\omega_0\rho}\cos\Omega = 0. \end{cases}$$

The stability properties of a constant solution are examined by applying the method of linearization. The eigenvalues polynomial of the Jacobin matrix is

$$\begin{vmatrix} -\mu + K\sin(\omega_0 T - \varphi) - \lambda & \dfrac{f}{4\omega_0}\cos\Omega \\ -3\omega_0\rho - \dfrac{f}{4\omega_0\rho^2}\cos\Omega & -\dfrac{f}{4\omega_0\rho}\sin\Omega - \lambda \end{vmatrix} = 0.$$

In terms of (8), we have

$$\lambda^2 - 2P\lambda + P^2 - Q = 0,$$

where $P = -\mu + K\sin(\omega_0 T - \varphi)$ and

$$Q = \{-3\omega_0\rho - \frac{1}{\rho^2}[-\rho\sigma + K\cos(\omega_0 T - \varphi) + \frac{3}{2}\omega_0\rho^3]\} \times [-\rho\sigma + K\cos(\omega_0 T - \varphi) + \frac{3}{2}\omega_0\rho^3].$$

The real parts of the eigenvalues are all negative if $P < 0,\ P^2 - Q > 0$, and then the constant solution is stable. Thus we find the stable conditions of constant solution:

$$\mu - K\sin(\omega_0 T - \varphi) > 0,$$

$$[\mu - K\sin(\omega_0 T - \varphi)]^2 + \{3\omega_0\rho + \frac{1}{\rho^2}[-\rho\sigma + K\cos(\omega_0 T - \varphi) + \frac{3}{2}\omega_0\rho^3]\}\times$$

$$[-\rho\sigma + K\cos(\omega_0 T - \varphi) + \frac{3}{2}\omega_0\rho^3] > 0.$$

The real parts of the eigenvalues equal zero if $P = 0,\ Q < 0$, and then Hopf bifurcation occurs. The Hopf bifurcation condition is:

$$\mu - K\sin(\omega_0 T - \varphi) = 0,$$

$$\{3\omega_0\rho + \frac{1}{\rho^2}[-\rho\sigma + K\cos(\omega_0 T - \varphi) + \frac{3}{2}\omega_0\rho^3]\} \times [-\rho\sigma + K\cos(\omega_0 T - \varphi) + \frac{3}{2}\omega_0\rho^3] > 0.$$

Let $\frac{d\rho}{dT} = 0$ in (10). Then the time delay T_0 is found to meet

$$\tan(\omega_0 T_0 - \varphi) = \frac{\mu}{\sigma - \frac{3}{2}\omega_0\rho^2}. \tag{12}$$

Results of stability analysis for typical case are given in Figure 1, together with results obtained by the numerical integration of equation (1), for the uncontrolled system (Fig.1 A), the controlled system without time delay (Fig.1 B), the controlled system with time delay according to (12) (Fig.1 C).

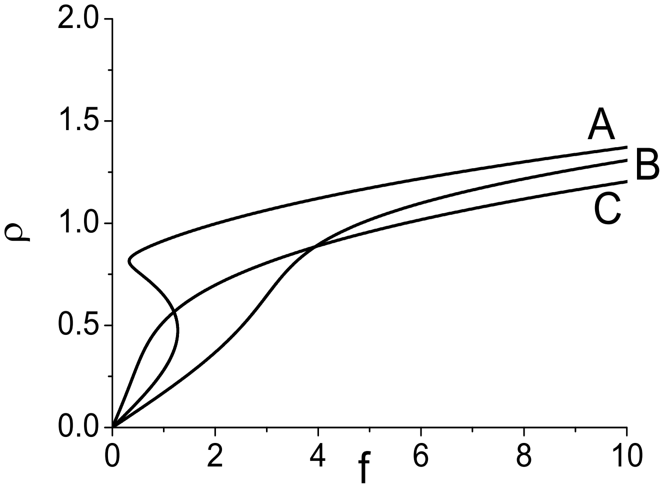

Fig.1 External excitation f - response ρ curves

Remark *The curves depicted in Fig. 1 are External excitation f - response ρ curves for the uncontrolled system (curve A), the controlled system with no time delay (curve B), and those with time delay (curve C) with parameters*

$$\omega = 1.1,\ \omega_0 = 1,\ \zeta = 0.01,\ \varepsilon = 0.1, \varphi = \frac{\pi}{2},\ K = 1,\ T = \frac{2\pi}{3}.$$

4 The condition of more solutions which amplitude ρ has

Let $M = \mu - K\sin(\omega_0 T - \varphi),\ N = \sigma - K\cos(\omega_0 T - \varphi)$. Then (10) can read

$$f = 4\omega_0\rho\sqrt{M^2 + (N - 1.5\omega_0\rho^2)^2}.$$

Set $\frac{df}{d\rho} = 0$. Then

$$A_1\rho^4 - A_2\rho^2 + A_3 = 0,$$

where

$$A_1 = 36\omega_0^3 + \frac{9}{4}\omega_0^2,\ A_2 = 24\omega_0^2 N + 3\omega_0 N,\ A_3 = M^2 + N^2.$$

Hence

$$\rho^2 = \frac{A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1}.$$

The condition of more solutions which amplitude ρ has is

$$f(\rho^2) < 0 \text{ or } f(\rho^2) > 0 \text{ i.e.}$$

$$M + N < 0 \text{ or } M + N > 0 \text{ i.e.}$$

$$\mu + \sigma - K(\sin(\omega_0 T - \varphi) + \cos(\omega_0 T - \varphi)) < 0 \text{ or}$$

$$\mu + \sigma - K(\sin(\omega_0 T - \varphi) + \cos(\omega_0 T - \varphi)) > 0.$$

Therefore, the condition of more solutions which amplitude ρ has can be written into

$$\mu + \sigma - \sqrt{2}K\cos(\omega_0 T - \varphi - \frac{\pi}{4}) < 0 \text{ or } \mu + \sigma - \sqrt{2}K\cos(\omega_0 T - \varphi - \frac{\pi}{4}) > 0.$$

5 Conclusion

We use the AP method and lowest order approximate solution to discuss the primary resonance of an externally excited Duffing oscillator under feedback control with time delay. We obtain the condition of constant solution, which is stable, the Hopf bifurcation condition and the condition of multi-solution to amplitude. Moreover, if the gains of delay feedback are small enough, the asymptotic perturbation method is a power tool to gain insight into the primary resonance of Duffing oscillator with weak non-linearity. Two slow-flow equations, governing the amplitude and phase of approximate long time response of the oscillator, have been derived and the external excitation-response curves have been given with numerical solutions. An appropriate stability analysis has been also performed and appropriate choices for the feedback gains and the time delay have been found in order to reduce the amplitude peak.

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