

A Note on the Rapid Convergence for the Quasilinear Elliptic Problem

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Abstract: In this paper, we obtain the result of k -th ($k \geq 2$) order convergence for the quasilinear elliptic problem via the variational approach.

Keywords: quasilinear elliptic boundary value problem; variational approach; generalized quasilinearization; k -th order convergence.

1 Introduction

The property of solutions for some elliptic problem has aroused a lot of attention, we can see the references and cited therein [1-4]. Recently, Lakshmikantham and Leela [5] obtained the monotone sequences that converge quadratically to the weak solution of the quasilinear elliptic problem by employing the generalized quasilinearization method. Motivated by the idea of Cabada, Nieto and Veiga [6], in this paper, we obtain a sequence of approximation weak solutions with the k -th ($k \geq 2$) order convergence via variational method for the quasilinear elliptic problem. For this purpose, the monotone iterative technique which was presented by Koksals and Lakshmikantham [7] is used in this study.

2 Preliminaries

We consider the quasilinear BVP

$$\begin{cases} -\Delta_p u + c(x)u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \text{ in the sense of trace,} \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the Sobolev spaces, Δ_p denote the p -Laplacian, that is, $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, Du denotes the gradient of u , $1 < p < n$. Assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function,

$c : \Omega \rightarrow \mathbb{R}_+$ is a measurable function with $c(x) > 0$ a.e. $x \in \Omega$ and $c \in L^1(\Omega)$.

We shall always mean that the boundary condition is in the sense of trace, so we shall not repeat it to avoid monotony.

Definition 2.1 (see [5]). The function $u \in W_0^{1,p}(\Omega)$ is said to be a weak solution of (1) if $f(x, u), f(x, u)u \in L^1(\Omega)$ and for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_{\Omega} |Du|^{p-2} Du Dv + cuv = \int_{\Omega} f(x, u)v.$$

Definition 2.2 (see [5]). The function $\alpha_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is said to be a weak lower solution of (1) if $\alpha_0(x) \leq 0$ on $\partial\Omega$, and

$$\int_{\Omega} |D\alpha_0|^{p-2} D\alpha_0 Dv + c\alpha_0 v \leq \int_{\Omega} f(x, \alpha_0)v$$

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is satisfied for each $v \in W_0^{1,p}(\Omega)$ with $v \geq 0$ a.e. in Ω . If the inequalities are reversed, then α_0 is said to be a weak upper solution of (1).

We need the following Lemmas and Corollary for our main result.

Lemma 2.1(see [5]). Let α_0, β_0 be weak lower and upper solutions of (1) respectively. Suppose further that

$$f(x, u_1) - f(x, u_2) \leq d(x)(u_1 - u_2) \quad \text{a.e. in } \Omega, \quad (2)$$

whenever $u_1 \geq u_2$ and $d : \Omega \rightarrow R_+$ is measurable, $d \in L^1(\Omega)$ and $c(x) - d(x) > 0$ a.e. in Ω . Then $\alpha_0(x) \leq \beta_0(x)$ a.e. in Ω .

Lemma 2.2. Assume that

(A₁) $\alpha_0, \beta_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ are weak lower and upper solutions of (1) such that $\alpha_0(x) \leq \beta_0(x)$ a.e. in Ω .

(A₂) $f \in \bar{\Omega} \times R \rightarrow R$ is a Caratheodory function, and is nondecreasing in u for $x \in \Omega$, a.e.;

(A₃) for any $\eta \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\alpha_0 \leq \eta \leq \beta_0(x)$, the function $h(x) = f(x, \eta) \in L^\delta(\Omega)$, $\delta > \frac{n}{p}$ and $0 < N \leq c(x)$ a.e. in Ω .

Then there exist monotone sequences $\{\alpha_n(x)\}, \{\beta_n(x)\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\alpha_n \rightarrow \rho, \beta_n \rightarrow r$ weakly in $W_0^{1,p}(\Omega)$ and ρ, r are minimal and maximal solutions of (1).

Lemma 2.2 is the same as Remark 3.1 in [7], so we omit the detail of this proof.

Corollary 2.1. In addition to the assumptions of Lemma 2.2, suppose that f satisfies the condition (2) of Lemma 2.1. Then $\rho = u = r$ is the unique weak solution of (1).

Lemma 2.3. In addition to the assumptions of Lemma 2.1, if, there exists a unique weak solution $u(x)$ of (1), then $\alpha_0(x) \leq u(x) \leq \beta_0(x)$ in Ω , a.e..

Proof. Since $u(x)$ is a unique weak solution of (1), we can consider $u(x)$ as a weak upper solution of (1) by Definition 2.1. In view of Lemma 2.1, we can get $\alpha_0(x) \leq u(x)$ in Ω , a.e.. Similarly, we can have $u(x) \leq \beta_0(x)$ in Ω , a.e.. This complete the proof.

To obtain the main result, We need the following inequality which holds for $p > 1$ and $u, v \in W_0^{1,p}(\Omega)$, namely

$$\int_{\Omega} [|Du|^{p-2}Du - |Dv|^{p-2}Dv](Du - Dv) \geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) \geq 0, \quad (3)$$

where $\|u\| = (\int_{\Omega} |Du|^p)^{1/p}$ is the equivalent norm for $W_0^{1,p}(\Omega)$ which is linked in a natural way with the p -Laplacian. For $p \geq 2$, the following stronger inequality holds, that is

$$\int_{\Omega} [|Du|^{p-2}Du - |Dv|^{p-2}Dv](Du - Dv) \geq c(p)\|u - v\|^p, \quad (4)$$

where $c(p) > 0$. The details of proof can be found in [5].

3 Main result

We are now in a position to describe the result of K -th order convergence for the quasilinear elliptic problem.

Theorem 3.1. Assume that

(A₁) $\alpha_0, \beta_0 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ are weak lower and upper solutions of (1) such that $\alpha_0(x) \leq \beta_0(x)$ a.e. in Ω ;

(A₂) $f \in \bar{\Omega} \times R \rightarrow R$ is a Caratheodory function, $\frac{\partial^i f}{\partial u^i} \geq 0$ exist and are Caratheodory functions, for $\alpha_0 \leq u \leq \beta_0$ a.e. in Ω , $i = 1, \dots, k$;

(A₃) There exist positive constants N, N_0 and M_1 such that $\frac{\partial^k f}{\partial u^k} \leq k!N_0$, and for any $u, u_1, u_2, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\alpha_0 \leq v \leq u \leq \beta_0, \alpha_0 \leq v \leq u_2 \leq u_1 \leq \beta_0$ such that

$$\sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{(u-v)^i}{i!} \in L^\delta(\Omega), \quad (u-v)^k \in W_0^{1,\delta}(\Omega), \quad c(x) - \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{(u-v)^{i-1}}{(i-1)!} \geq N$$

and

$$0 < \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{1}{i!} [(u_1 - v)^{i-1-j} (u_2 - v)^j] \leq M_1 < c(x)$$

a.e. in Ω , where $\delta > \frac{n}{p}$, $M_1 \in L^1(\Omega)$.

Then there exists a monotone sequence $\{\alpha_n\} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\alpha_n \rightarrow \rho$ weakly in $W_0^{1,p}(\Omega)$ and $\rho = u$ is a weak solution of (1) satisfying $\alpha_0 \leq u \leq \beta_0$ a.e. in Ω . Moreover the convergence is of order k ($k \geq 2$) for the case $p \geq 2$.

Proof. For given $x \in \Omega$, by using the Taylor expansion method, we obtain that

$$f(x, u) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{(u - v)^i}{i!} + \frac{\partial^k f}{\partial u^k}(x, \xi) \frac{(u - v)^k}{k!}, \tag{5}$$

with $\alpha_0 \leq w \leq \xi \leq u \leq \beta_0$. In view of (A₂), we have

$$f(x, u) \geq \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{(u - v)^i}{i!}.$$

Thus, let

$$g(x, u, v) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{(u - v)^i}{i!}. \tag{6}$$

Clearly, $g(x, u, v) \in L^\delta(\Omega)$ and

$$g(x, u, v) \leq f(x, u), \quad g(x, u, u) = f(x, u). \tag{7}$$

Also,

$$g_u(x, u, v) = \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{(u - v)^{i-1}}{(i - 1)!} \geq 0.$$

By using that for all $A, B \in R$, $A^i - B^i = (A - B) \sum_{j=0}^{i-1} A^{i-1-j} B^j$ and (A₃), we have

$$\begin{aligned} g(x, u_1, v) - g(x, u_2, v) &= \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{1}{i!} [(u_1 - v)^i - (u_2 - v)^i] \\ &= \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(x, v) \frac{1}{i!} \sum_{j=0}^{i-1} [(u_1 - v)^{i-1-j} (u_2 - v)^j] (u_1 - u_2) \\ &\leq M_1 (u_1 - u_2). \end{aligned}$$

We consider the problem

$$\begin{cases} -\Delta_p u + c(x)u = g(x, u, \alpha_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

In view of (A₁) and (7), we have

$$\begin{cases} \int_{\Omega} |D\alpha_0|^{p-2} D\alpha_0 Dv + c\alpha_0 v \leq \int_{\Omega} f(x, \alpha_0)v = \int_{\Omega} g(x, \alpha_0, \alpha_0)v & \text{in } \Omega, \\ \alpha_0 \leq 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \int_{\Omega} |D\beta_0|^{p-2} D\beta_0 Dv + c\beta_0 v \geq \int_{\Omega} f(x, \beta_0)v dx \geq \int_{\Omega} g(x, \beta_0, \alpha_0)v dx & \text{in } \Omega, \\ \beta_0 \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, α_0, β_0 are weak lower and upper solutions of (8) respectively. By Lemma 2.1 and Lemma 2.3, we obtain that there exists a unique weak solution $\alpha_1 \in W_0^{1,p}(\Omega)$ of (8) provided $c(x) - M_1 > 0$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$.

Then consider the problem

$$\begin{cases} -\Delta_p u + c(x)u = g(x, u, \alpha_1) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (9)$$

Using (A₁), (7), (8) and (9), we have

$$\begin{cases} \int_{\Omega} |D\alpha_1|^{p-2} D\alpha_1 Dv + c\alpha_1 v \leq \int_{\Omega} f(x, \alpha_1)v = \int_{\Omega} g(x, \alpha_1, \alpha_1)v & \text{in } \Omega, \\ \alpha_1 = 0 & \text{on } \partial\Omega \end{cases} \quad (10)$$

and

$$\begin{cases} \int_{\Omega} |D\beta_0|^{p-2} D\beta_0 Dv + c\beta_0 v \geq \int_{\Omega} f(x, \beta_0)v dx \geq \int_{\Omega} g(x, \beta_0, \alpha_1)v dx & \text{in } \Omega, \\ \beta_0 \geq 0 & \text{on } \partial\Omega. \end{cases} \quad (11)$$

Hence, α_1, β_0 are weak lower and upper solution of (9) respectively. By Lemma 2.1 and Lemma 2.3, we obtain that there exists a unique weak solution $\alpha_2 \in W_0^{1,p}(\Omega)$ of (9) provided $c(x) - M_1 > 0$ such that $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \beta_0$. Using this procedure successively, we can get a monotone sequence of solutions $\{\alpha_n\}$ satisfying

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n \leq \alpha_{n+1} \leq \beta_0,$$

where the element α_{n+1} of the sequence is a weak solution of the problem

$$\begin{cases} -\Delta_p u + c(x)u = g(x, u, \alpha_n) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, $\{\alpha_n\}$ is a nondecreasing sequence. By the monotone character of $\{\alpha_n\}$ there exists pointwise limit $\lim_{n \rightarrow \infty} \alpha_n(x) = \rho(x)$ a.e. in Ω . Moreover, since $\alpha_0 \leq \alpha_n \leq \beta_0$ a.e. in Ω , it follows by dominated convergence theorem that $\alpha_n(x) \rightarrow \rho(x)$ in $L^p(\Omega)$. We note that α_n satisfies

$$\int_{\Omega} |D\alpha_n|^{p-2} D\alpha_n Dv + c\alpha_n v = \int_{\Omega} g(x, \alpha_n, \alpha_{n-1})v,$$

for each $v \in W_0^{1,p}(\Omega)$, $v \geq 0$, a.e. Using (A₃) and (3) with $v = \alpha_n$, we obtain

$$\|\alpha_n\|^p + N \int_{\Omega} |\alpha_n|^2 \leq \int_{\Omega} g(x, \alpha_n, \alpha_{n-1})\alpha_n.$$

Since by (A₃), the right-hand side belongs to $L^\delta(\Omega)$, this implies that $\sup_n \|\alpha_n\|_{W_0^{1,p}(\Omega)} < \infty$. Therefore, there exists a subsequence $\{\alpha_{n_j}\}$ which converges weakly in $W_0^{1,p}(\Omega)$ to ρ .

To check that ρ is a weak solution of (1), we fix $v \in W_0^{1,p}(\Omega)$ and see that

$$\int_{\Omega} |D\alpha_n|^{p-2} D\alpha_n Dv + c\alpha_n v = \int_{\Omega} g(x, \alpha_n, \alpha_{n-1})v. \quad (12)$$

Let $Q(Du) = |Du|^{p-2} Du$ and by (3), Q satisfies

$$\int_{\Omega} [Q(D\alpha_n) - Q(Dw)][D\alpha_n - Dw] \geq 0 \quad (13)$$

for each $w \in W^{1,p}(\Omega)$, $\alpha_n \neq w$. Since $\{Q(D\alpha_n)\}$ is bounded in $L^p(\Omega)$, $Q(D\alpha_n) \rightarrow \xi$ weakly in $L^p(\Omega)$ for some $\xi \in L^p(\Omega)$. Taking the limit as $n \rightarrow \infty$ in (12), we have

$$\int_{\Omega} \xi Dv + c\rho v = \int_{\Omega} f(x, \rho)v. \quad (14)$$

Setting $v = \alpha_n$ in (12) and substituting (13), we have

$$\int_{\Omega} g(x, \alpha_n, \alpha_{n-1})\alpha_n - c\alpha_n^2 - Q(D\alpha_n)Dw - Q(Dw)(D\alpha_n - Dw) \geq 0.$$

As $n \rightarrow \infty$, it follows that

$$\int_{\Omega} f(x, \rho)\rho - c\rho^2 - \xi Dw - Q(Dw)(D\alpha_n - Dw) \geq 0. \tag{15}$$

Setting $v = \rho$ in (14), then by using (14) and (15), we arrive at

$$\int_{\Omega} \xi D\rho - \xi Dw - Q(Dw)[D(\rho - w)] \geq 0$$

which leads to

$$\int_{\Omega} [\xi - Q(Dw)]D(\rho - w) \geq 0.$$

Choose $w = \rho - \lambda v$, $\lambda > 0$, so that one has

$$\int_{\Omega} [\xi - Q(D(\rho - \lambda v))]Dv \geq 0.$$

As $\lambda \rightarrow 0$, there results

$$\int_{\Omega} [\xi - Q(D\rho)]Dv \geq 0.$$

Setting $v = -v$, we see $\int_{\Omega} [\xi - Q(D\rho)]Dv \leq 0$ and thus we get

$$\int_{\Omega} [\xi - Q(D\rho)]Dv = 0.$$

Consequently,

$$\int_{\Omega} \xi Dv = \int_{\Omega} Q(D\rho)Dv = \int_{\Omega} [f(x, \rho) - c\rho]v$$

or equivalently,

$$\int_{\Omega} Q(D\rho)Dv + c\rho v = \int_{\Omega} f(x, \rho)v$$

for $v \in W_0^{1,p}(\Omega)$, $v \geq 0$, so we get ρ is a weak solution of (1) by definition $Q(D\rho)$.

Finally, we prove k -th order convergence of the sequence $\{\alpha_n\}$ to ρ . Let $P_{n+1} = \rho - \alpha_{n+1}$ and note that $P_{n+1}(0) = 0$ on $\partial\Omega$. For each $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e., we have

$$\begin{aligned} \int_{\Omega} [|D\rho|^{p-2}D\rho Dv - |D\alpha_{n+1}|^{p-2}D\alpha_{n+1}]Dv + cP_{n+1}v &= \int_{\Omega} [f(x, \rho) - g(x, \alpha_{n+1}, \alpha_n)]v \\ &= \int_{\Omega} \left[\frac{\partial^k}{\partial u^k} f(x, \xi) \frac{(\rho - \alpha_n)^k}{k!} + g_u(x, \sigma, \alpha_n)P_{n+1} \right] v, \end{aligned}$$

where $\alpha_{n+1} \leq \sigma \leq \rho$. Hence, we get

$$\int_{\Omega} [|D\rho|^{p-2}D\rho - |D\alpha_{n+1}|^{p-2}D\alpha_{n+1}]Dv + c_0(x)P_{n+1}v = \int_{\Omega} \left[\frac{\partial^k}{\partial u^k} f(x, \xi) \frac{(\rho - \alpha_n)^k}{k!} \right] v,$$

where $c_0(x) = c(x) - g_u(x, \sigma, \alpha_n) \geq N > 0$.

Taking $v = P_{n+1}$ and using (A₃), we obtain

$$\int_{\Omega} [|D\rho|^{p-2}D\rho - |D\alpha_{n+1}|^{p-2}D\alpha_{n+1}]DP_{n+1} + NP_{n+1}^2 \leq N_0 \int_{\Omega} P_n^k P_{n+1}.$$

If $p \geq 2$, we can use the estimate (4) to get

$$c(p) \|P_{n+1}\|^p \leq N_0 \|P_n^k\|_{L^{np/n(p-1)+p}(\Omega)} \|P_{n+1}\|_{L^{p^*}(\Omega)} \leq \tilde{N}_0 \|P_n^k\|_{L^\delta(\Omega)} \|P_{n+1}\|$$

for suitable constant \tilde{N}_0 . It then follows that

$$\|p_{n+1}\|^{p-1} \leq N^0 \|p_n^k\|_{W_0^{1,\delta}(\Omega)},$$

where N^0 is a suitable constant. The proof is therefore completed.

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