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The Periodicity Character of a Difference Equation

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Abstract: In this paper we investigate the boundedness and the periodicity character of solutions of a max-type difference equation with periodic coefficients of period three. We prove that every solution of the equation under consideration is periodic of period three.

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1 Introduction

Our purpose in this paper is to study the boundedness and the periodicity character of solutions of the difference equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}, \ n = 0,1,... \tag{1}$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive numbers of period three and $A_n \in (0,1]$ for all $n = 0,1,...$ such that the elements of one of the three subsequences $\{A_{3i}\}_{i=0}^{\infty}$, $\{A_{3i+1}\}_{i=0}^{\infty}$ or $\{A_{3i+2}\}_{i=0}^{\infty}$ equal one.

Eq.(1) have been studied by many authors. In [2], it was established for Eq.(1) with the sequence $\{A_n\}_{n=0}^{\infty}$ is identically equal to a constant number that every solution is eventually periodic with period two. If the sequence $\{A_n\}_{n=0}^{\infty}$ is periodic with period two or three the authors in [3] and [4], respectively proved that every solution of Eq.(1) is periodic with period two.

In [6], the authors proved that every solution of Eq.(1) is periodic with period three provided that the sequence $\{A_n\}_{n=0}^{\infty}$ is periodic with period two.

Also, for some difference equations with the property that every solutions eventually periodic, see [8] and [9] where the authors proved under some conditions that every solution of the difference equation

$$x_{n+1} = \sum_{i=1}^{k} \frac{A_i}{x_{n-i}}, \quad n = 0,1,2,...$$

is periodic with period p where p can be formulated in terms of the coefficients $A_0, A_1, ..., A_{k-1}$. See also [1,5,7,10-16]

We now present some definitions and known results which will be useful in our investigation of Eq.(1).

Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}), \quad n = 0,1,2,... \tag{2}$$

with initial conditions $y_{-k}, y_{-k+1}, ..., y_0 \in I$, where I is some interval of real numbers.

We say that $\overline{y}$ is an equilibrium point of Eq.(2) if

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$$f(\overline{y},\overline{y},...,\overline{y})=\overline{y}.$$

That is, the constant sequence $\{y_n\}_{n=-k}^{\infty}$ with

$$y_n=\overline{y} \quad \text{for all} \ \ n\geq -k,$$

is a solution of Eq.(2).

Thus Eq.(1) has a unique equilibrium point $\overline{x}=1$.

We give some basic definitions about semi-cycles of solutions of Eq.(1).

Definition 1.1 *A positive semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(1) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to the equilibrium $\overline{x}$ with $l\geq -1$ and $m\leq\infty$, such that*

either $l=-1$, or $l>-1$ and $x_{l-1}<\overline{x}$

and

either $m=\infty$, or $m<\infty$ and $x_{m+1}<\overline{x}$.

Definition 1.2 *A negative semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(1) consists of a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than the equilibrium $\overline{x}$, with $l\geq -1$ and $m\leq\infty$, such that*

either $l=-1$, or $l>-1$ and $x_{l-1}\geq\overline{x}$

and

either $m=\infty$, or $m<\infty$ and $x_{m+1}\geq\overline{x}$.

Definition 1.3 *A sequence $\{x_n\}_{n=-1}^{\infty}$ is called non-oscillatory if there exists $N\geq -1$ such that*

either $x_n>\overline{x}$ for all $n\geq N$ or $x_n<\overline{x}$ for all $n\geq N$.

Definition 1.4 *A sequence $\{x_n\}_{n=-1}^{\infty}$ is called oscillatory if it is not non-oscillatory.*

For more informations about the semi-cycles of Eq.(1) we refer to the article [12].

In the sequel we assume that $\{A_n\}_{n=0}^{\infty}=\{\alpha,\beta,1,\alpha,\beta,1,...\}$ with $\alpha>\beta$.

2 Boundedness of solutions

In this section we show that every solutions of Eq.(1) is bounded and persists.

Theorem 2.1 *Every positive solution of Eq.(1) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1). First we claim that $\{x_n\}_{n=-1}^{\infty}$ is bounded below if and only if it is bounded above. In the following we prove this claim:

Suppose that $\{x_n\}_{n=-1}^{\infty}$ is bounded from above by a positive number, say M, we shall show that $\{x_n\}_{n=-1}^{\infty}$ is bounded below. Now it follows from Eq.(1) that

$$x_{n+1}=\max\left\{\frac{1}{x_n},\frac{A_n}{x_{n-1}}\right\}\geq\max\left\{\frac{1}{M},\frac{A_n}{M}\right\}\geq\frac{\beta}{M}:=m.$$

Conversely Suppose that $\{x_n\}_{n=-1}^{\infty}$ is bounded from below by a positive number, say m, we shall show that $\{x_n\}_{n=-1}^{\infty}$ is bounded above. Again it follows from Eq.(1) that

$$x_{n+1}=\max\left\{\frac{1}{x_n},\frac{A_n}{x_{n-1}}\right\}\leq\max\left\{\frac{1}{m},\frac{A_n}{m}\right\}=\frac{1}{m}: \text{ and this proves the claim.}$$

Now it suffices to show that $\{x_n\}_{n=-1}^{\infty}$ is bounded from above. For the sake of contradiction, suppose that it is not true that $\{x_n\}_{n=-1}^{\infty}$ is bounded from above. Then there exists $N>0$ such that

$$\max\{x_n:-1\leq n\leq N\}<x_N.$$

Hence it follows that there exists integers N_1 and N_2 with $N < N_2 < N_1$, such that

$$x_{N_1} \leq \min\{x_n : -1 \leq n \leq N_1\},$$

and

$$x_{N_2} = \max\{x_n : -1 \leq n \leq N_1\}.$$

Thus

$$x_{N_1} = \max\left\{\frac{1}{x_{N_1-1}}, \frac{A_{N_1-1}}{x_{N_1-2}}\right\} \geq \max\left\{\frac{1}{x_{N_2}}, \frac{A_{N_1-1}}{x_{N_2}}\right\} = \frac{1}{x_{N_2}}.$$

Then

$$x_{N_1}x_{N_2} \geq 1.$$

Also,

$$x_{N_2} = \max\left\{\frac{1}{x_{N_2-1}}, \frac{A_{N_2-1}}{x_{N_2-2}}\right\} < \max\left\{\frac{1}{x_{N_1}}, \frac{A_{N_2-1}}{x_{N_1}}\right\} = \frac{1}{x_{N_1}}.$$

Then

$$x_{N_1}x_{N_2} < 1,$$

which is a contradiction, and so the proof of the theorem is complete. ■

Theorem 2.2 *Every solution of Eq.(1) which is bounded below by* $\sqrt{\alpha}$ *enters the interval* $\left[\sqrt{\alpha}, \frac{1}{\sqrt{\alpha}}\right]$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1) and there exists $N \geq 0$ such that

$$x_{n-1} \geq \sqrt{\alpha} \quad \text{for all} \;\; n \geq N.$$

It follows from Eq.(1) that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} \leq \max\left\{\frac{1}{\sqrt{\alpha}}, \frac{A_N}{\sqrt{\alpha}}\right\} = \frac{1}{\sqrt{\alpha}}.$$

Similarly, we see that

$$x_{N+2} = \max\left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} \leq \max\left\{\frac{1}{\sqrt{\alpha}}, \frac{A_{N+1}}{\sqrt{\alpha}}\right\} = \frac{1}{\sqrt{\alpha}}.$$

Similarly to the above then

$$x_{n-1} \in \left[\sqrt{\alpha}, \frac{1}{\sqrt{\alpha}}\right] \quad \text{for all} \;\; n \geq N.$$

The proof complete. ■

3 Analysis of the semi-cycle of Eq.(1)

The following lemmas are quiet important results concerning the analysis of the semi-cycle of Eq.(1). These results shall be used in the sequel. For the proof of these lemmas we refer to the article [4].

Lemma 3.1 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(1) which is not eventually constant. Then* $x_n \neq 1$ *for all* $n \geq 1$.

Lemma 3.2 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(1), and let* $m \geq 0$ *be a non-negative integer. Then one of the following statements is true.*

1. $x_{m-1}x_m = 1$.
2. $x_m x_{m+1} = 1$.
3. $x_{m+1}x_{m+2} = 1$.
4. $x_{m+2}x_{m+3} = 1$.

Lemma 3.3 *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1) which is not eventually constant. Then the following statements are true.*

1. $\{x_n\}_{n=-1}^{\infty}$ oscillates about the positive equilibrium point $\overline{x} = 1$ of Eq.(1).

2. With the possible exception of the first negative semi-cycle, every negative semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length equal to one.

3. Let $n \geq 1$ be such that $x_{n-2} \geq 1$ and $x_{n-1} < 1$. Then $x_n = \dfrac{1}{x_{n-1}}$.

4. Every positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length at most two.

Lemma 3.4 *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1) which is not eventually constant. Then the following statements are true.*

1. With the possible exception of the first positive semi-cycle, every positive semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has a strict maximum which occurs in the first term of the positive semi-cycle.

2. With the possible exception of the first positive semi-cycle, the first term of every positive semi-cycle is less than or equal to the last term of the preceding positive semi-cycle.

4 The Main Result

In this section we show that every positive solution of Eq.(1) is periodic solution of period three.

Theorem 4.1 *Eq.(1) has a periodic solution of period three.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1). Suppose there exists $N \geq 0$ such that

$$x_{N-1} > \overline{x} > x_N.$$

Then from Eq.(1), we see that

$$x_{N+1} = \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} = \frac{1}{x_N}.$$

Choose $\{A_{3i+1}\}_{i=0}^{\infty} = \{1,1,1,...\}$, it follows from Eq.(1) that

$$\begin{aligned}
x_{N+2} &= \max\left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} = \max\left\{x_N, \frac{1}{x_N}\right\} = \frac{1}{x_N}, \\
x_{N+3} &= \max\left\{\frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}}\right\} = \max\left\{x_N, A_{N+2}x_N\right\} = x_N, \\
x_{N+4} &= \max\left\{\frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}}\right\} = \max\left\{\frac{1}{x_N}, A_N x_N\right\} = \frac{1}{x_N}, \\
x_{N+5} &= \max\left\{\frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}}\right\} = \max\left\{x_N, \frac{1}{x_N}\right\} = \frac{1}{x_N}, \\
x_{N+6} &= \max\left\{\frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}}\right\} = \max\left\{x_N, A_{N+2}x_N\right\} = x_N, \\
x_{N+7} &= \max\left\{\frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}}\right\} = \max\left\{\frac{1}{x_N}, A_N x_N\right\} = \frac{1}{x_N}, \\
x_{N+8} &= \max\left\{\frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}}\right\} = \max\left\{x_N, \frac{1}{x_N}\right\} = \frac{1}{x_N},
\end{aligned}$$

and

$$x_{N+9} = \max\left\{\frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}}\right\} = \max\left\{x_N, A_{N+2}x_N\right\} = x_N.$$

Then it follows by induction that $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution of period three of the form

$$\left\{..., x_N, \frac{1}{x_N}, \frac{1}{x_N}, x_N, \frac{1}{x_N}, \frac{1}{x_N}, ...\right\}.$$

■

Theorem 4.2 *Every positive solution of Eq.(1) which is bounded below by $\sqrt{\alpha}$ is eventually periodic with period three.*

Proof. Since the positive semi-cycle has length at most two and the negative semi-cycle is of length exactly one, we consider only the following cases for an integer $N \geq 0$

(a) $x_N < \overline{x} < x_{N-1}$.

(b) $x_{N-1} < \overline{x} < x_N$.

(c) $x_{N-1},\ x_N \geq \overline{x}$.

We will prove Case (a) (the other cases are similar and the proof will be omitted).
Assume (a) holds, then we see from Eq.(1) that

$$\begin{aligned} x_{N+1} &= \max\left\{\frac{1}{x_N}, \frac{A_N}{x_{N-1}}\right\} = \max\left\{\frac{1}{x_N}, \frac{\alpha}{x_{N-1}}\right\} = \frac{1}{x_N}, \\ x_{N+2} &= \max\left\{\frac{1}{x_{N+1}}, \frac{A_{N+1}}{x_N}\right\} = \max\left\{x_N, \frac{\beta}{x_N}\right\} = x_N, \end{aligned}$$

where $x_N > \sqrt{\alpha} > \sqrt{\beta} \ \Rightarrow x_N^2 > \beta \ \Rightarrow \ x_N > \dfrac{\beta}{x_N}$,

$$\begin{aligned} x_{N+3} &= \max\left\{\frac{1}{x_{N+2}}, \frac{A_{N+2}}{x_{N+1}}\right\} = \max\left\{\frac{1}{x_N}, x_N\right\} = \frac{1}{x_N}, \\ x_{N+4} &= \max\left\{\frac{1}{x_{N+3}}, \frac{A_{N+3}}{x_{N+2}}\right\} = \max\left\{x_N, \frac{\alpha}{x_N}\right\} = x_N, \\ x_{N+5} &= \max\left\{\frac{1}{x_{N+4}}, \frac{A_{N+4}}{x_{N+3}}\right\} = \max\left\{\frac{1}{x_N}, \beta x_N\right\} = \frac{1}{x_N}, \\ x_{N+6} &= \max\left\{\frac{1}{x_{N+5}}, \frac{A_{N+5}}{x_{N+4}}\right\} = \max\left\{x_N, \frac{1}{x_N}\right\} = \frac{1}{x_N}, \\ x_{N+7} &= \max\left\{\frac{1}{x_{N+6}}, \frac{A_{N+6}}{x_{N+5}}\right\} = \max\left\{x_N, \alpha x_N\right\} = x_N, \\ x_{N+8} &= \max\left\{\frac{1}{x_{N+7}}, \frac{A_{N+7}}{x_{N+6}}\right\} = \max\left\{\frac{1}{x_N}, \beta x_N\right\} = \frac{1}{x_N}, \end{aligned}$$

and

$$x_{N+9} = \max\left\{\frac{1}{x_{N+8}}, \frac{A_{N+8}}{x_{N+7}}\right\} = \max\left\{x_N, \frac{1}{x_N}\right\} = \frac{1}{x_N}.$$

In this case we see that the solution becomes in the form

$$\left\{..., \frac{1}{x_N}, x_N, \frac{1}{x_N}, \frac{1}{x_N}, x_N, \frac{1}{x_N}, ...\right\},$$

and so the solution is periodic with period three. This completes the proof. ■

Theorem 4.3 *Assume $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(1) and suppose there exists $m \geq 1$ such that*

$$x_m < \sqrt{\alpha}.$$

Then either $\{x_n\}_{n=-1}^{\infty}$ is eventually periodic solution with period three or

$$\liminf_{n\to\infty} x_n \geq \sqrt{\alpha}.$$

Proof. Observe from Eq.(1) that $x_m x_{m+1} \geq 1 \ \Rightarrow \ x_{m+1} \geq \dfrac{1}{x_m} > \dfrac{1}{\sqrt{\alpha}} \ \Rightarrow \ \alpha x_{m+1} > \sqrt{\alpha} > x_m$

$$\begin{aligned} x_{m+2} &= \max\left\{\frac{1}{x_{m+1}}, \frac{A_{m+1}}{x_m}\right\} = \max\left\{\frac{1}{x_{m+1}}, \frac{\alpha}{x_m}\right\} = \frac{\alpha}{x_m}, \\ x_{m+3} &= \max\left\{\frac{1}{x_{m+2}}, \frac{A_{m+2}}{x_{m+1}}\right\} = \max\left\{\frac{x_m}{\alpha}, \frac{\beta}{x_{m+1}}\right\} = \frac{x_m}{\alpha}, \end{aligned}$$

and

$$x_{m+4} = \max\left\{\frac{1}{x_{m+3}}, \frac{A_{m+3}}{x_{m+2}}\right\} = \max\left\{\frac{\alpha}{x_m}, \frac{x_m}{\alpha}\right\}.$$

We consider the following two cases

(A1) $x_{m+4} = \frac{x_m}{\alpha}$. In this case $\frac{x_m}{\alpha} > \frac{\alpha}{x_m}$, and we see that

$$\begin{aligned} x_{m+5} &= \max\left\{\frac{1}{x_{m+4}}, \frac{A_{m+4}}{x_{m+3}}\right\} = \max\left\{\frac{\alpha}{x_m}, \frac{\alpha^2}{x_m}\right\} = \frac{\alpha}{x_m}, \\ x_{m+6} &= \max\left\{\frac{1}{x_{m+5}}, \frac{A_{m+5}}{x_{m+4}}\right\} = \max\left\{\frac{x_m}{\alpha}, \frac{\alpha\beta}{x_m}\right\} = \frac{x_m}{\alpha}, \end{aligned}$$

where $\frac{x_m}{\alpha} > \frac{\alpha}{x_m},\ \beta < 1\ \Rightarrow\ \frac{x_m}{\alpha} > \frac{\alpha}{x_m} > \frac{\alpha\beta}{x_m}$,

$$\begin{aligned} x_{m+7} &= \max\left\{\frac{1}{x_{m+6}}, \frac{A_{m+6}}{x_{m+5}}\right\} = \max\left\{\frac{\alpha}{x_m}, \frac{x_m}{\alpha}\right\} = \frac{x_m}{\alpha}, \\ x_{m+8} &= \max\left\{\frac{1}{x_{m+7}}, \frac{A_{m+7}}{x_{m+6}}\right\} = \max\left\{\frac{\alpha}{x_m}, \frac{\alpha^2}{x_m}\right\} = \frac{\alpha}{x_m}, \end{aligned}$$

and

$$x_{m+9} = \max\left\{\frac{1}{x_{m+8}}, \frac{A_{m+8}}{x_{m+7}}\right\} = \max\left\{\frac{x_m}{\alpha}, \frac{\alpha\beta}{x_m}\right\} = \frac{x_m}{\alpha}.$$

In this case we see that the solution is in the form

$$\left\{..., \frac{\alpha}{x_m}, \frac{x_m}{\alpha}, \frac{x_m}{\alpha}, \frac{\alpha}{x_m}, \frac{x_m}{\alpha}, \frac{x_m}{\alpha}, ...\right\}.$$

Therefore $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution with period three.

(A2) $x_{m+4} = \frac{\alpha}{x_m}$. In this case $\frac{\alpha}{x_m} > \frac{x_m}{\alpha}$, and we see that

$$x_{m+5} = \max\left\{\frac{1}{x_{m+4}}, \frac{A_{m+4}}{x_{m+3}}\right\} = \max\left\{\frac{x_m}{\alpha}, \frac{\alpha^2}{x_m}\right\}.$$

We consider the following two cases

(B1) $x_{m+5} = \frac{x_m}{\alpha}$. In this case $\frac{x_m}{\alpha} > \frac{\alpha^2}{x_m}$, and we see that

$$x_{m+6} = \max\left\{\frac{1}{x_{m+5}}, \frac{A_{m+5}}{x_{m+4}}\right\} = \max\left\{\frac{\alpha}{x_m}, \frac{\beta x_m}{\alpha}\right\} = \frac{\alpha}{x_m},$$

where $\beta < 1,\ \frac{\alpha}{x_m} > \frac{x_m}{\alpha}\ \Rightarrow\ \frac{\alpha}{x_m} > \frac{x_m}{\alpha} > \frac{\beta x_m}{\alpha}$,

$$\begin{aligned} x_{m+7} &= \max\left\{\frac{1}{x_{m+6}}, \frac{A_{m+6}}{x_{m+5}}\right\} = \max\left\{\frac{x_m}{\alpha}, \frac{\alpha}{x_m}\right\} = \frac{\alpha}{x_m}, \\ x_{m+8} &= \max\left\{\frac{1}{x_{m+7}}, \frac{A_{m+7}}{x_{m+6}}\right\} = \max\left\{\frac{x_m}{\alpha}, x_m\right\} = \frac{x_m}{\alpha}, \end{aligned}$$

and

$$x_{m+9} = \max\left\{\frac{1}{x_{m+8}}, \frac{A_{m+8}}{x_{m+7}}\right\} = \max\left\{\frac{\alpha}{x_m}, \frac{\beta x_m}{\alpha}\right\} = \frac{\alpha}{x_m}.$$

In this case we see that the solution is in the form

$$\left\{..., \frac{x_m}{\alpha}, \frac{\alpha}{x_m}, \frac{\alpha}{x_m}, \frac{x_m}{\alpha}, \frac{\alpha}{x_m}, \frac{\alpha}{x_m}, ...\right\}.$$

Therefore $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution with period three.

(B2) $x_{m+5} = \dfrac{\alpha^2}{x_m}$. In this case $\dfrac{\alpha^2}{x_m} > \dfrac{x_m}{\alpha}$, and we see that

$$x_{m+6} = \max\left\{\frac{1}{x_{m+5}}, \frac{A_{m+5}}{x_{m+4}}\right\} = \max\left\{\frac{x_m}{\alpha^2}, \frac{\beta x_m}{\alpha}\right\} = \frac{x_m}{\alpha^2},$$

and in this case we see that

$$x_m < x_{m+3} = \frac{x_m}{\alpha} < x_{m+6} = \frac{x_m}{\alpha^2}$$

Thus

$$\liminf_{n\to\infty} x_n \geq \sqrt{\alpha}.$$

■

Remark 4.4 *Observe by Lemma 3.4 that the case where*

$$x_m, x_{m+1} < \sqrt{\alpha} \quad \textit{for} \quad m \geq 2$$

does not exist.

Theorem 4.2 and Theorem 4.3 lead to the following main result in this section.

Theorem 4.5 *Every solution of Eq.(1) is periodic with period three.*

In the following lemma we show that it is important that every element of at least one of the subsequences $\{A_{3i}\}$, $\{A_{3i+1}\}$ or $\{A_{3i+2}\}$ equal to one for the existence of period three solutions.

Lemma 4.1 *Assume that* $A_n \in (0,1)$ *for all* $n = 0, 1, 2, \ldots$. *Then Eq.(1) has no periodic solution of period three.*

Proof. For the sake of contradiction, assume that there exists a periodic solution of period three $\{\ldots, \alpha, \beta, \gamma, \alpha, \beta, \gamma, \ldots\}$. Therefore we see from Eq.(1) that

$$\alpha = \max\left\{\frac{1}{\gamma}, \frac{A_3}{\beta}\right\},$$
$$\beta = \max\left\{\frac{1}{\alpha}, \frac{A_1}{\gamma}\right\},$$

and

$$\gamma = \max\left\{\frac{1}{\beta}, \frac{A_2}{\alpha}\right\}.$$

We study the following different possibilities

(1) $\alpha = \frac{1}{\gamma}$, $\beta = \frac{1}{\alpha}$ and $\gamma = \frac{1}{\beta}$. Then $\alpha = \beta = \gamma = 1$ (a contradiction).

(2) $\alpha = \frac{1}{\gamma}$, $\beta = \frac{1}{\alpha}$ and $\gamma = \frac{A_2}{\alpha}$. Then $\alpha\gamma = A_2 = 1$ (a contradiction).

(3) $\alpha = \frac{1}{\gamma}$, $\beta = \frac{A_1}{\gamma}$ and $\gamma = \frac{1}{\beta}$. Then $\beta\gamma = A_1 = 1$ (a contradiction).

(4) $\alpha = \frac{1}{\gamma}$, $\beta = \frac{A_1}{\gamma}$ and $\gamma = \frac{A_2}{\alpha}$. Then $\alpha\gamma = A_2 = 1$ (a contradiction).

(5) $\alpha = \frac{A_3}{\beta}$, $\beta = \frac{1}{\alpha}$ and $\gamma = \frac{1}{\beta}$. Then $\alpha\beta = A_3 = 1$ (a contradiction).

(6) $\alpha = \frac{A_3}{\beta}$, $\beta = \frac{1}{\alpha}$ and $\gamma = \frac{A_2}{\alpha}$. Then $\alpha\beta = A_3 = 1$ (a contradiction).

(7) $\alpha = \frac{A_3}{\beta}$, $\beta = \frac{A_1}{\gamma}$ and $\gamma = \frac{1}{\beta}$. Then $\beta\gamma = A_1 = 1$ (a contradiction).

(8) $\alpha = \frac{A_3}{\beta}$, $\beta = \frac{A_1}{\gamma}$ and $\gamma = \frac{A_2}{\alpha}$. Then $\alpha\beta = A_3$, $\beta\gamma = A_1$, $\alpha\gamma = A_2$.

Then $\alpha = \sqrt{\frac{A_2A_3}{A_1}}$, $\beta = \sqrt{\frac{A_1A_3}{A_2}}$, $\gamma = \sqrt{\frac{A_1A_2}{A_3}}$.

Since

$$\gamma = \max\left\{\frac{1}{\beta}, \frac{A_2}{\alpha}\right\} = \frac{A_2}{\alpha},$$

it follows that

$$\frac{A_2}{\alpha} > \frac{1}{\beta},$$

and so

$$A_1 > 1 \quad \text{(a contradiction)}.$$

From the previous computations we see that Eq.(1) has no periodic solution of period three. ■

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