

Homotopy Analysis of Stretching Flows with Partial Slip

M. Sajid *

Theoretical Plasma Physics Division, PINSTECH, P.O. Nilore,
Islamabad 44000, Pakistan

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Abstract: The investigation deals with the slip effects on the two problems of viscous incompressible flows induced by a stretching sheet. These flow problems corresponds to the planar and axisymmetric stretching. After a similarity transformation, The arising system has been solved by homotopy analysis method (HAM). Convergence of the HAM solution is checked. Comparison of the HAM results corresponding to the planar stretching problem is provided with that of the existing results. The influence of the slip parameter on the flow is explored.

Keywords: two-dimensional flow; axisymmetric flow; Partial slip; stretching sheet; HAM solution

1 Introduction

Boundary layer flow induced by a stretching sheet is important in many engineering processes. Such flows are of interest in the manufacture of sheeting material through an extrusion process, glass fibre drawing, crystal growing, polymer industries etc. Specifically, the radial stretching occurs during the expansion of balloons. Sakiadis [1,2] presented the pioneering work in this area. Crane [3] presented a closed form exponential solution for the planar viscous flow of linear stretching case. Later this problem has been extended to various aspects by considering non-Newtonian fluids, more general stretching velocity, magnetohydrodynamic (MHD) effects, porous sheets, porous media and heat or mass transfer. Some relevant studies are mentioned in the references [4-13]. In all these investigation no-slip condition is used. But in the situation when the fluid is particulate such as emulsions, suspensions, foams and polymer solutions, the no-slip condition is inadequate. In such cases the suitable boundary condition is the partial slip. In spite of its importance in polymer and electrochemical industry, no proper attention has been given to the flow analysis with partial slip condition. Wang [14] discussed the partial slip effects on the planar stretching flow. He obtained the perturbation and numerical solutions.

The objective of the present study is two fold. Firstly to obtain the HAM solution for the planar stretching flow problem considered by Wang [14]. Secondly to construct the HAM solution for the radial stretching flow with partial slip. The focus of attention is to obtain the flow velocity. The arrangement of the paper is in the following.

The problem is formulated in section 2. Section 3 contains the HAM [15] solution. The HAM is powerful mathematical technique and has been already applied to several non-linear problems [16-32]. Section 4 includes the analysis of results. The concluding remarks are given in section 5.

2 Mathematical formulation

This section provides the derivation of an axisymmetric flow. For the derivation of two-dimensional flow (in planar case) the readers are referred to ref. [14]. Consider the steady, laminar flow of a viscous fluid over a

* **Corresponding author.** Tel.: +92 51 2207274, Fax: +92 51 2207256. E-mail address: sajidqau2002@yahoo.com

stretching sheet (at $z = 0$) and the fluid occupies the space $z > 0$. We choose cylindrical polar coordinate system (r, θ, z) and flow under the rotational symmetry. All the physical quantities are taken independent upon θ and the azimuthal component of velocity v vanishes identically. The equations which govern the axisymmetric flow are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right), \tag{2}$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right), \tag{3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, ρ is the density, μ is the dynamic viscosity, p is the pressure and u and w are the velocities in r and z directions respectively. Since the flow is only due to the stretching of the sheet therefore the pressure gradient can be neglected. The boundary layer flow is governed through Eq. (1) and

$$u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2}. \tag{4}$$

The partial slip boundary conditions are

$$\begin{aligned} u &= ar + k\nu \frac{\partial u}{\partial z}, & w &= 0 & \text{at } z &= 0, \\ u &\rightarrow 0 & & & \text{as } z &\rightarrow \infty. \end{aligned} \tag{5}$$

in which $a > 0$ is the stretching constant and k is a constant of proportionality.

To solve Eqs. (4) and (5), we introduce the non-dimensional parameters η and velocity f which are defined as

$$\eta = \sqrt{\frac{a}{\nu}} z, \quad u = ar f'(\eta), \quad w = -2\sqrt{a\nu} f(\eta) \tag{6}$$

Applying the above transformations, Eqs. (4) and (5) lead to the following non-dimensional problem

$$f''' + 2ff'' - f'^2 = 0, \tag{7}$$

$$f(0) = 0, \quad f'(0) = 1 + K f''(0), \quad f'(\infty) = 0. \tag{8}$$

where

$$K = k\sqrt{a\nu}$$

The equation which governs the two-dimensional flow (in the planar stretching case) is given by [14]

$$f''' + ff'' - f'^2 = 0 \tag{9}$$

with the conditions (8). We combine Eqs. (7) and (9) in the following form

$$f''' + mf f'' - f'^2 = 0, \tag{10}$$

where m is a constant and takes the value 1 for planar flow and 2 for the axisymmetric flow. In next section we will solve Eq. (10) subject to conditions (8) using HAM.

3 HAM solution

3.1 Zeroth-order deformation problem

The velocity $f(\eta)$ can be expressed by the set of base functions

$$\left\{ \eta^k \exp(-n\eta) \mid k \geq 0, n \geq 0 \right\} \tag{11}$$

in the form of the following series

$$f(\eta) = a_{0,0}^0 + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-n\eta), \quad (12)$$

in which $a_{m,n}^k$ are the coefficients. Invoking the so-called *Rule of solution expressions* for $f(\eta)$ and Eq. (8) the initial guess $f_0(\eta)$ and linear operators \mathcal{L} are

$$f_0(\eta) = \frac{1}{K+1} [1 - \exp(-\eta)], \quad (13)$$

$$\mathcal{L}(f) = f''' - f', \quad (14)$$

where

$$\mathcal{L}[C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3] = 0, \quad (15)$$

and $C_1 - C_3$ are the constants. Equation (10) show that the nonlinear operator is:

$$\mathcal{N}[\bar{f}(\eta, p)] = \frac{\partial^3 \bar{f}(\eta, p)}{\partial \eta^3} - \left(\frac{\partial \bar{f}(\eta, p)}{\partial \eta} \right)^2 + m f \frac{\partial^2 \bar{f}(\eta, p)}{\partial \eta^2}, \quad (16)$$

If \hbar is the auxiliary nonzero parameter then the zeroth-order deformation problem is

$$(1-p) \mathcal{L}[\bar{f}(\eta, p) - f_0(\eta)] = p \hbar \mathcal{N}[\bar{f}(\eta, p)], \quad (17)$$

$$\bar{f}(0, p) = 0, \quad \bar{f}'(0, p) = 1 + K \bar{f}''(0, p), \quad \bar{f}'(\infty, p) = 0, \quad (18)$$

where $p \in [0, 1]$ is an embedding parameter. For $p = 0$ and $p = 1$, we have

$$\bar{f}(\eta, 0) = f_0(\eta), \quad \bar{f}(\eta, 1) = f(\eta, \xi). \quad (19)$$

The initial guess $f_0(\eta)$ tends to $f(\eta, \xi)$ as p varies from 0 to 1. By Taylor's series expansion:

$$\bar{f}(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \quad (20)$$

where

$$f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \bar{f}(\eta, p)}{\partial p^m} \right|_{p=0}, \quad (21)$$

and the convergence of the series (20) depends upon the values of the parameter \hbar . The value of \hbar is chosen in such a way that the series (20) is convergent at $p = 1$. Then by using Eq. (19) one obtains

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (22)$$

3.2 m th-order deformation problems

Here we first differentiate Eq. (17) m times with respect to p then divide by $m!$ and setting $p = 0$ we obtain

$$\mathcal{L}[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar \mathcal{R}_m(\eta), \quad (23)$$

$$f_m(0) = f_m'(0) - K f_m''(0) = f_m'(\infty) = 0, \quad (24)$$

where

$$\mathcal{R}_m(\eta) = \left[\frac{\partial^3 f_{m-1}}{\partial \eta^3} - \sum_{k=0}^{m-1} f'_{m-1-k} f'_k + m \sum_{k=0}^{m-1} f_{m-1-k} f''_k \right], \quad (25)$$

The general solutions of Eq. (23) – (25) can be written as

$$f_m(\eta) = f_m^*(\eta) + C_1 \exp(-\eta) + C_2 \exp(\eta) + C_3, \tag{26}$$

where $f_m^*(\eta)$ are the particular solutions and the constants are determined by the boundary conditions (24) which are given by

$$C_2 = 0, \quad C_1 = \frac{1}{K+1} \left[\frac{\partial f_m^*(\eta)}{\partial \eta} \Big|_{\eta=0} - K \frac{\partial^2 f_m^*(\eta)}{\partial \eta^2} \Big|_{\eta=0} \right], \quad C_3 = -C_1 - f_m^*(0, \xi), \tag{27}$$

In the next section, the linear non-homogeneous Eqs. (23) – (25) are solved using Mathematica in the order $m = 1, 2, 3, \dots$

4 Results analysis

The explicit, analytic expression given in Eq. (22) is the series solution of the problem. The convergence region and rate of approximation is dependent upon the choice of the value of the auxiliary parameter \hbar for the homotopy analysis method. In Figs. 1 and 2 the \hbar -curves are plotted for four different values of the slip parameter K for both planar and axisymmetric flows. Obviously that the range for the admissible \hbar -values is $-2.0 \leq \hbar < 0$. Besides that the series converges for all values of K for both the flows when $\hbar = -1$.

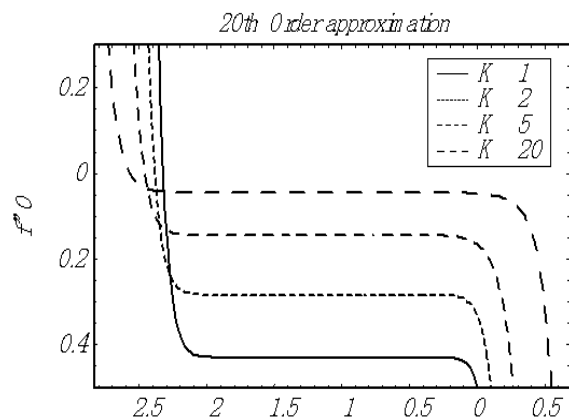


Figure 1: \hbar -curves for different values of K for the planar case.

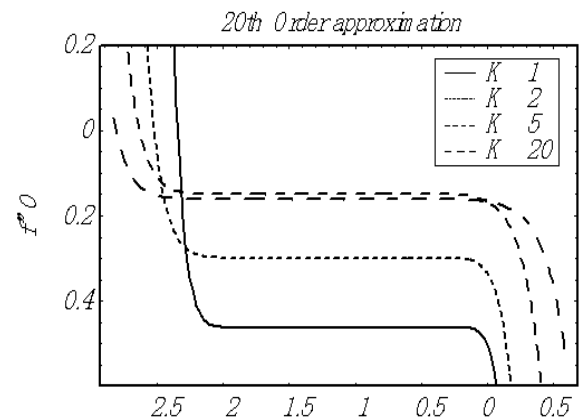


Figure 2: \hbar -curves for different values of K for the axisymmetric case.

To see the effect of slip parameter on the velocity in both flows Figs. 3 – 6 have been displayed. It is shown in Figs. 3 – 6 that the velocity components decreases with an increase in the slip parameter for both planar and axisymmetric flows. Figs. 7 and 8 have been plotted just to see how the velocity behaves when we consider both type of flows. It is evident that velocity is less for the axisymmetric flow when compared with the planar case. The values of $f''(0)$ and $f(\infty)$ are given in tables 1 and 2 for both flow situations. The results presented in Table 1 and Figs. 3 and 4 are the same as presented by Wang [14] which shows that the HAM results are agreeing well with those of already presented in the literature.

Table 1 Values of $f''(0)$ and $f(\infty)$ in the case of planar flow.

K	0	0.3	1	2	5	20
$f''(0)$	-1	-0.701548	-0.430160	-0.283980	-0.144840	-0.043790
$f(\infty)$	1	0.888558	0.754878	0.657298	0.525166	0.352698

Table 2 Values of $f''(0)$ and $f(\infty)$ in the case of axisymmetric flow.

K	0	0.3	1	2	5	20
$f''(0)$	-1.173720	-0.784705	-0.462510	-0.299050	-0.149392	-0.044370
$f(\infty)$	0.751503	0.657115	0.550951	0.476417	0.378072	0.251887

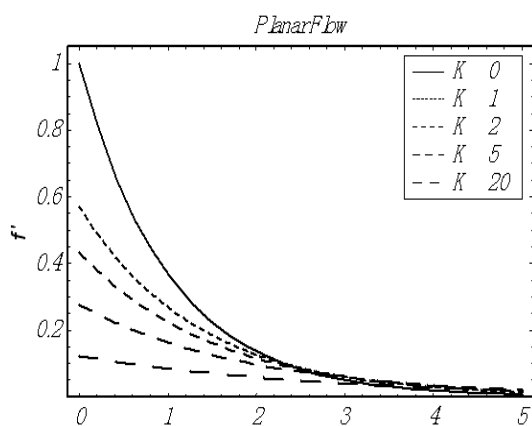


Figure 3: Variation of f' by increasing the slip parameter K for planar flow.

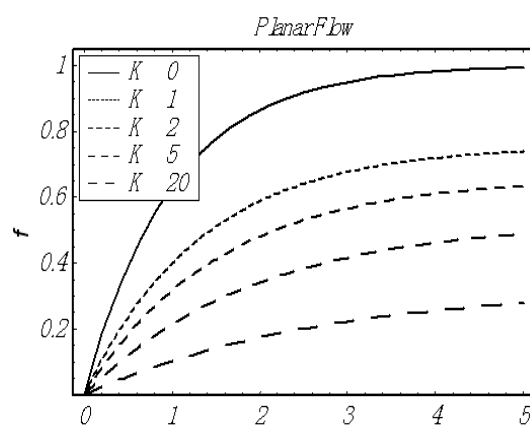


Figure 4: Variation of f by increasing the slip parameter K for planar flow.

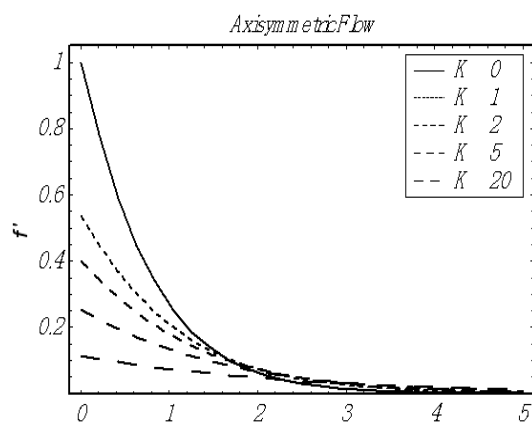


Figure 5: Variation of f' by increasing the slip parameter K for axisymmetric flow.

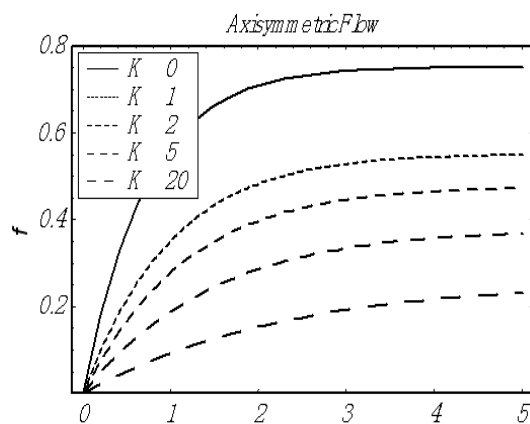


Figure 6: Variation of f by increasing the slip parameter K for axisymmetric flow.

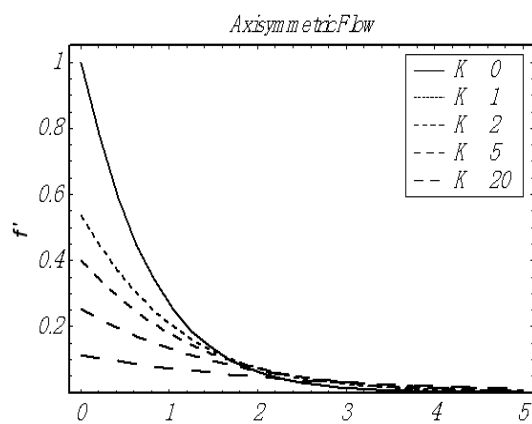


Figure 7: Variation of f' for planar and axisymmetric flow.

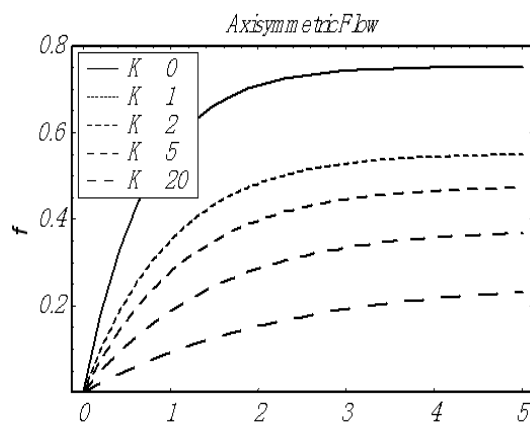


Figure 8: Variation of f for planar and axisymmetric flow.

5 Concluding remarks

In this paper, the analysis is carried out to obtain the series solutions for the planar and axisymmetric flows in a viscous fluid with partial slip condition. The analytic solution is presented using homotopy analysis method and the convergence of the series is properly discussed. The effects of the slip parameter on the velocity are presented graphically and discussed. The values of $f''(0)$ and $f(\infty)$ are tabulated and found in excellent agreement with those in ref. [14].

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