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Existence and Uniqueness of Solution for P-Laplacian Dirichlet Problem

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Abstract: This paper deals with equation $-\Delta_p u(x) + \lambda|u(x)|^{p-2} = f(x, u(x))$ in bounded smooth domain $\Omega \subset R^N$ with Dirichlet boundary value condition. The existence results are obtained by Browder Theorem and uniqueness of solution is also considered.

Keywords: the p-Laplacian BV problems; uniqueness of solution; Dirichlet boundary condition

AMS Subject Classification: 35J60, 35B30, 35B40

1 Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u(x) + \lambda|u(x)|^{p-2}u(x) = f(x, u(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

where Δ_p is the p-Laplacian, $\Omega \in C^{0,1}$ be a bounded domain in R^N. Let $p \geq 2$, $\lambda > 0$ and $f : \Omega \times R \longrightarrow R$ be a caratheodory function which is decreasing with respect to the second variable, i.e.,

$$f(x, s_1) \geq f(x, s_2) \quad for\ a.a.\ x \in \Omega\ \ and s_1, s_2 \in R, \quad s_1 \leq s_2 \tag{2}$$

Assume, moreover, that there exists $f_0 \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$ and $c > 0$ such that

$$|f(x, s)| \leq f_0(x) + c|s|^{p-1} \tag{3}$$

We considered such problems with numerical methods in [2-5]

Definition 1 *We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to (1) if*
$\int |\nabla u|^{p-2}\nabla u \nabla v + \lambda \int |u(x)|^{p-2}u(x)v(x) = \int f(x, u(x))\, v(x)dx$ *for all* $v \in W_0^{1,p}(\Omega)$.

Definition 2 *Let H be a real Hilbert space. An operator $T : H \to H$ satisfying*

$$(T(u) - T(v), u - v) \geq 0 \tag{4}$$

for any $u, v \in H$ is called a monotone operator. An operator T is called strictly monotone if for $u \neq v$ the strict inequality holds in (4). An operator T is called strongly monotone if there exists $c > 0$ such that
$(T(u) - T(v), u - v) \geq c \parallel u - v \parallel^2$ *for any* $u, v \in H$.

Our main results concerning problem (1) is the following:

Theorem 3 *Let $\Omega \in C^{0,1}$ be a bounded domain in R^N and $p \geq 2$, $\lambda > 0$ and $f \in CAR(\Omega \times R)$ satisfy (2) and (3). Then problem (1) has a unique weak solution.*

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2 Proof of theorem 3

We consider the Sobolev space $W_0^{1,p}(\Omega)$ with the norm (see [1])

$$\| u \|= (\int |\nabla u(x)|^p dx)^{\frac{1}{p}}.$$

Let us recall the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}$ where $p^* = \frac{Np}{N-p}$ and

the compact embedding

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$$

where $q \in [1, \frac{Np}{N-p})$.

We define for $\lambda \in R$ operators $J, G, F : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ by

$$< J(u), v >= \int |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx$$

$$< G(u), v >= \int |u(x)|^{p-2} u(x) v(x) dx$$

$$< F(u), v >= \int f(x, u(x))\; v(x) dx$$

for all $u, v \in W_0^{1,p}(\Omega)$. It follows from (4) that J, G, F are well defined.

Set $T := J + \lambda G - F$. We say that u is a weak solution of (1) if

$$< Tu, v >=< J(u), v > +\lambda < G(u), v > - < F(u), v >= 0$$

holds for any $v \in W_0^{1,p}(\Omega)$. Thus, to find a weak solution of (1) is equivalent to finding $u \in W_0^{1,p}(\Omega)$ which satisfies the operator equation $T(u) = 0$ (see [6]). We have the following properties of J, G, F:

a) J, G and F are bounded operators in the sense that they map bounded sets onto bounded sets. Indeed $\forall u \;\; \| u \|_{W_0^{1,p}(\Omega)} \leq M$ we have:

$$\|J(u)\|_{(W_0^{1,p}(\Omega))^*} = sup_{\|v\| \leq 1} |< J(u), v >|$$
$$\leq sup_{\|v\| \leq 1} (\int | \nabla u |^{(p-1)p')})^{\frac{1}{p'}} \;\; (\int |\nabla v|^p)^{\frac{1}{p}} \leq \|u\|^{\frac{p}{p'}} \leq M^{\frac{p}{p'}}$$

$$\|G(u)\|_{(W_0^{1,p}(\Omega))^*} = sup_{\|v\| \leq 1} |< G(u), v >|$$
$$\leq sup_{\|v\| \leq 1} (\int | u |^{(p-1)p')})^{\frac{1}{p'}} \;\; (\int |v|^p)^{\frac{1}{p}} \leq c\|u\|^{\frac{p}{p'}} \leq cM^{\frac{p}{p'}}$$

$$\|F(u)\|_{(W_0^{1,p}(\Omega))^*} = sup_{\|v\| \leq 1} |< F(u), v >| \leq (\int |f|^{p'})^{\frac{1}{p'}} \;\; (\int |v|^p)^{\frac{1}{p}} \leq c_{emb} \| f \|_{p'}$$

where c_{emb} is the constant of the embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$.

b) J, G and F are continuous operators. If $u_n \to u$ in $W_0^{1,p}(\Omega)$ we have:

$$\|J(u_n) - J(u)\|_{(W_0^{1,p}(\Omega))^*} = sup_{\|v\| \leq 1} |< J(u_n) - J(u), v >|$$
$$= sup_{\|v\| \leq 1} | \int (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla v dx |$$
$$\leq sup_{\|v\| \leq 1} (\int | |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u|^{p'} \; dx \,)^{\frac{1}{p'}} (\int | \nabla v |^p \; dx)^{\frac{1}{p}}$$
$$\leq c_{emb} (\int | |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \; |^{p'} \; dx)^{\frac{1}{p'}} \;\; \to 0$$

$$\|G(u_n) - G(u)\|_{(W_0^{1,p}(\Omega))^*} \leq (\int | \, |u_n|^{p-2} u_n - |u|^{p-2} u \; |)^{p'} dx)^{\frac{1}{p'}} \; c_{emb} \to 0$$

$$\|F(u_n)-F(u)\|_{(W_0^{1,p}(\Omega))^*} \le sup_{\|v\|\le 1}(\int (f(x,u_n)-f(x,u))vdx$$
$$\le c_{emb}(\int \mid (f(x,u_n)-f(x,u)\mid^{p'} dx)^{\frac{1}{p'}} \to 0.$$

c) Let $p \ge 2$, then for all $x_1, x_2 \in R^N$ we have the following inequality(see [8])

$$|x_2|^p \ge |x_1|^p + p|x_1|^{p-2}x_1(x_2-x_1) + \frac{|x_2-x_1|^p}{2^{p-1}-1} \tag{5}$$

$$< J(u)-J(v), u-v >= \int (|\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x))(\nabla u(x)-\nabla v(x))dx$$
$$= \int |\nabla u(x)|^{p-2}\nabla u(x)(\nabla v(x)-\nabla v(x))dx$$
$$- \int |\nabla v(x)|^{p-2}\nabla v(x)(\nabla v(x)-\nabla v(x))dx$$
$$= I_1 + I_2$$

by using (5), we have

$$I_1 + I_2 \ge \frac{2}{p(2^{p-1}-1)}\int |\nabla u - \nabla v|^p dx = c\|u-v\|^p$$

So

$$< J(u)-J(v), u-v >\ge c\|u-v\|^p \quad for \quad p \ge 2 \tag{6}$$

$$< G(u)-G(v), u-v >= \int |u(x)|^{p-2}u(x)(u(x)-v(x)) - |v(x)|^{p-2}v(x)(v(x)-u(x))dx$$
$$= I_3 + I_4$$
$$I_3 + I_4 \ge \frac{2}{p(2^{p-1}-1)}\|u-v\|^p$$

then

$$< G(u)-G(v), u-v >\ge 0 \tag{7}$$

also

$$< F(v)-F(u), u-v >= \int (f(x,v)-f(x,u))(u-v)$$

Since f is decreasing with respect to the second variable, we have

$$\int (f(x,v)-f(x,u))(u-v) \ge 0,$$

consequently

$$< F(v)-F(u), u-v >\ge 0. \tag{8}$$

(6), (7), (8) imply that

$$< Tu - Tv, u-v >\ge c\|u-v\|^p, \tag{9}$$

so T is strongly monotone. (see e.g., [9])

d) Indeed T being strongly monotone implies

$$< Tu, u >\ge< T0, u > +c\|u\|^p$$

on the other hand

$$< T0, u >=< J0, u > +\lambda < G0, u > - < F0, u >$$
$$= -\int f(x,0)udx \ge -(\int f(x,0)^{p'})^{\frac{1}{p'}}(\int |u|^p)^{\frac{1}{p}}$$
$$\ge -Mc_{emb}\|u\|$$

then

$$< Tu, u > \geq c\|u\|^p - Mc_{emb}\|u\|.$$

So

$$\lim_{\|u\| \to \infty} \frac{< T(u), u >}{\| u \|} = \infty$$

hence T is coercive. The existence and uniqueness of weak solution for problem (1) follows from following theorem. Thus Theorem 3 is proved.

Theorem 4 ((Browder) [7]) *: Let X be a reflexive real Banach space.Moreover let $T : X \to X^*$ be an operator satisfying the conditions*
i) *T is bounded.*

ii) *T is demicontinuous, i.e., T maps strongly convergent sequence in X to weakly convergent sequences in X^*.*

iii) *T is coercive, i.e.,*

$$\lim_{\|u\| \to \infty} \frac{< T(u), u >}{\| u \|} = \infty$$

iv) *T is monotone on the space X, i.e., for all $u, v \in X$ we have*

$$< T(u) - T(v), u - v > \geq 0 \tag{10}$$

then the equation $T(u) = f^*$ has at least one solution $u \in X$ for every $f^* \in X^*$.(If moreover, the inequality (10) is strict for all $u, v \in X,\ u \neq v$), then the equation (10) has precisely one solution $u \in X$ for every $f^* \in X^*$.

The uniqueness of solution, is also a direct consequence of (9). Suppose that u, v be solutions of problem (1) such that $u \neq v$ we have :

$$0 =< Tu - Tv, u - v > \geq c\|u - v\|^p \geq 0$$

therefore $u = v$.

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