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Single Traveling Wave Solutions to $\alpha - \beta$ Family Equation

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Abstract: Under the wave travelling transformation, $\alpha - \beta$ family equation is reduced to an ordinary differential equation. Using a symmetry group of one parameter, this ODE is reduced to a second-order linear inho-mogeneous ODE. Furthermore, we apply the change of the variable and complete discrimination system for polynomlal to solve the corresponding integrals and obtain a number of single traveling wave solutions to $\alpha - \beta$ family equation.

Keywords: traveling wave solutions; symmetry group; $\alpha - \beta$ Family equation

1 Introduction

In [1], Degasperis and procesi introduced the following family of third order dispersive PDE conservation laws:

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xxx})_x, \tag{1}$$

where $\alpha, \gamma, c, i = 0, 1, 2, 3$ are real constants. They found that only three equations that satisfy the asymptotic integrability condition within this family :the Korteweg-de Vries equation, the Camassa-holm equation and the Degasperis-Procesi equation. Learning from this, in this paper we will study the $\alpha - \beta$ Family equation:

$$u_t - \varepsilon u_{xxt} + k u_x + \gamma u_{xxx} + \alpha u u_x = \varepsilon(u u_{xxx} + \beta u_x u_{xx}), \tag{2}$$

where k, α, β are constants. Similar to equation (1), we can get many important equations from (2).

When $\varepsilon = 1, \alpha = -6, \beta = 0, \gamma = 0, k = 0$, (2) becomes the well-known Korteweg-de Vries equation

$$u_t - 6uu_x + uu_{xxx} = 0, \tag{3}$$

which has solitary wave solutions and its solitary waves are solitude [2].

For $\varepsilon = 1, \alpha = 3, \beta = 2, \gamma = 0, k = 0$, (2) becomes Camass-Holm equation:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{4}$$

which describes the unidirectional propagation of shallow water waves over a flat bottom ,it has peakons, cuspons, stumpons, composite wave solutions [3]. It also has compactons [4].

With $\varepsilon = 1, \alpha = 4, \beta = 3, \gamma = 0, k = 0$,we find Degasperis-Procesi equation:

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{5}$$

has a multitude of peculiar wave solutions: peakons, cuspons, composite waves, and stumpons [5].

For $\varepsilon = 1, \alpha = 1, \beta = 2, \gamma = 0, k = 1$, it is Fornberg-Whitham equation:

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{6}$$

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This equation is derived by B.Fornberg and G.B Whitham,where u is the fluid velovity in the direction. In [6] Fornberg and Whitham obtained a peaked solution of the form$u(x,t) = A\exp(-\frac{1}{2}\left|x-\frac{4t}{3}\right|)$, where A is an arbitrary constant. In([7],[8]), Jiangbo Zhou and Lixin Tian constructed two types of bounded traveling wave solutions, which were called kink-like and antikink-like wave solutions. They obtained the implicit expression for solutions and the explicit expression for peakons and periodic cusp wave solutions. At the same time , they showed that the limits of solutions and periodic cusp wave solutions were peakons. Though the form of the Fornberg-Whitham equation looks very similar to the Degasperis-Procesi equation, there are great differences between them . The Fornberg-Whitham equation is nonintegrable.

In the past decade, a lot of expansion methods have been proposed to seek for the traveling wave solutions to nonlinear partial differential equations . These methods are only indirect methods based on the some assumptions about the forms of the solutions of the equations considered. Applying those indirect methods, we can't give all single traveling wave solutions to the equations considered. On the other hand, some superficially different solutions are essentially the same solution . So it is worthwhile to give the classifications of all single traveling wave solutions to those equations. However, direct integral method is a rather simple and powerful methods([9]-[12]).

In the present paper, we give the complete classification of classifical traveling wave solutions of $\alpha-\beta$ Family equation. Using the trick of symmetry group to reduce the $\alpha-\beta$ Family equation to an integrable ODE, and furthermore using the change of variable and complete discrimination system for polynomial to solve the corresponding integtral, we obtain the classification of all single traveling wave solutions to $\alpha-\beta$ Family equation. The method and trick used here have general meaning for the studies of traveling wave solutions to some nonlinear partial differenet equation.

2 The generation form of solution of $\alpha-\beta$ Family equation

Under the traveling wave transformation $u=u(\xi),\xi=x-ct$,$\alpha-\beta$ Family equation is reduced to the following ODE

$$(k-c)u'+\alpha uu'+\varepsilon(cu'''-uu''-\beta u'u'')+\gamma u'''=0. \tag{7}$$

Integrating once yields the following equation

$$(u-c-\frac{\gamma}{\varepsilon})u''=\frac{(k-c)}{\varepsilon}u+\frac{\alpha}{2\varepsilon}u^2-\frac{\beta}{2}(u')^2. \tag{8}$$

We take the change of variable as follows:

$$v=u-c-\frac{\gamma}{\varepsilon}. \tag{9}$$

Then Eq.(8) becomes

$$vv''=\frac{(k-c)}{\varepsilon}(v+c+\frac{\gamma}{\varepsilon})+\frac{\alpha}{2\varepsilon}(v+c+\frac{\gamma}{\varepsilon})^2-\frac{\beta}{2}(v')^2. \tag{10}$$

Since the Eq.(2) is invariant under the following one parameter group

$$u^*=u,\xi^*=u+\varepsilon. \tag{11}$$

So we can reduce the order of the Eq.(8) by one . For the purpose , we take $z=v^{'}$, then we have $v^{''}=z\frac{dz}{dv}$. Instituting those terms into the Eq.(8) and furthermore letting $w=z^2$, we have

$$w^{'}=-\frac{\beta}{v}w+(d_2v+d_1+d_0v^{-1}), \tag{12}$$

where $d_2=\frac{\gamma}{\varepsilon}$,$d_1=\frac{2}{\varepsilon}(k-c+\alpha c+\frac{\alpha\gamma}{\varepsilon})$,$d_0=\frac{1}{\varepsilon}[2(k-c)(c+\frac{\gamma}{\varepsilon})+\alpha(c+\frac{\gamma}{\varepsilon})^2]$. Using the method of the variation of constants, the general solution of the Eq.(12) is as follows :

$$w(v)=\frac{d_2}{\beta+2}v^2+\frac{d_1}{\beta+1}v+\frac{d_0}{\beta}+c_0v^{-\beta}, \tag{13}$$

which can be also written as

$$\int \frac{v^{\frac{\beta}{2}}dv}{\sqrt{\frac{d_2}{\beta+2}v^{2+\beta}+\frac{d_1}{\beta+1}v^{1+\beta}+\frac{d_0}{\beta}v^{\beta}+c_0}}=\pm(\xi-\xi_0), \tag{14}$$

where c_0 is an arbitrary constant. Which is the general form of solution of ODE (7). If we can derive the solutions of Eq. (14), the traveling wave soltions of $\alpha-\beta$ Family equation are obtained. In section 3 we will study this problem by means of complete discrimination system for polynomial and direct integral method. The key steps are to change the given $\alpha-\beta$ Family equation into the integral from like Eq.(14) and to analyze the solutions of Eq.(14).

The following important equation comes from Eq.(2). When $\varepsilon=1,\alpha=1,\beta=3,k=1,\gamma=0$,Eq.(2) becomes Eq.(6).

Under traveling wave transformation Eq.(7), Eq.(6) becomes

$$(1-c)u'+uu'+cuu'''-uu'''-3u'u''=0, \tag{15}$$

which is easily reduced to the following form from Eq.(12)

$$w'=-\frac{3}{v}w+[2+(2c-c^2)v^{-1}], \tag{16}$$

The general form of solution of Eq.(6) is

$$\int \frac{dv}{\sqrt{\frac{1}{2}v+\frac{(2c-c^2)}{3}+c_0v^{-3}}}=\pm(\xi-\xi_0), \tag{17}$$

when $\varepsilon=1,\alpha=4,\beta=3,k=0,\gamma=0$,Eq.(2) becomes Eq.(5).

Under traveling wave transformation Eq.(7), Eq.(5) becomes

$$-cu'+4uu'+cu'''-uu''-3u'u''=0, \tag{18}$$

which is easily reduced to the following form from Eq.(12)

$$w^{'}=-\frac{3}{v}w+3c+2c^2v^{-1}, \tag{19}$$

The general form of solution of Eq.(5) is

$$\int \frac{dv}{\sqrt{\frac{3c}{2}v+\frac{2c^2}{3}+c_0v^{-3}}}=\pm(\xi-\xi_0), \tag{20}$$

when $\varepsilon=1,\alpha=4,\beta=3,k=0,\gamma=0$,Eq.(2) becomes Eq.(4).

Under traveling wave transformation Eq.(7), Eq.(4) becomes

$$-cu'+3uu'+cu'''-uu''-2u'u''=0, \tag{21}$$

which is easily reduced to the following form from Eq.(12)

$$w^{'}=-\frac{3}{v}w+3c+c^2v^{-1}. \tag{22}$$

The general form of solution of Eq.(4) is

$$\int \frac{dv}{\sqrt{\frac{4c}{3}v+\frac{c^2}{3}+c_0v^{-2}}}=\pm(\xi-\xi_0), \tag{23}$$

when $\varepsilon=1,\alpha=-6,\beta=0,k=0,\gamma=0$, Eq.(2) becomes Eq.(3).

Under traveling wave transformation Eq.(7), Eq.(4) becomes

$$-cu' - 6uu' + cu''' = 0, \tag{24}$$

which is easily reduced to the following form from Eq.(12)

$$w' = -\frac{7}{2} - 8c^2 v^{-1}. \tag{25}$$

The general form of solution of Eq.(4) is

$$\int \frac{dv}{\sqrt{\frac{4c}{3}v + \frac{c^2}{3} + c_0 v^{-2}}} = \pm(\xi - \xi_0). \tag{26}$$

3 Single traveling wave solutions to $\alpha - \beta$ Family equation

When $\beta = 1$,then Eq.(14) becomes

$$\int \frac{v^{\frac{1}{2}} dv}{\sqrt{a_3 v^3 + a_2 v^2 + a_1 v + a_0}} = \pm(\xi - \xi_0), \tag{27}$$

where $\alpha_3 = \frac{d_2}{3}$, $a_2 = \frac{d_1}{2}$, $a_1 = d_0$, a_0 is an arbitrary constant. For simplicity ,we take the change of variable $U = a_3^{\frac{1}{3}} v$, then Eq.(27) becomes

$$\int \frac{U^{\frac{1}{2}} dv}{\sqrt{U^3 + t_2 U^2 + t_1 U + t_0}} = \pm a_3^{\frac{1}{2}} (\xi - \xi_0), \tag{28}$$

where $t_2 = -a_3^{-\frac{2}{3}} a_2$, $t_1 = a_3^{-\frac{1}{3}} a_1$, $t_0 = a_0$.

Denote $F(U) = U^3 + t_2 U^2 + t_1 U + t_0$, its complete discrimination system is $\Delta = -27(\frac{2t_2^2}{27} + t_0 - \frac{t_1 t_2}{3})^2 - 4(t_1 - \frac{t_2^2}{3})^3$ and $D_1 = t_1 - \frac{t_2^2}{3}$.

There are the following four cases to be discussed.

Case 1: $\Delta = 0$, $D_1 < 0$, then we have $F(U) = (U - \lambda)^2 (U - \gamma)$, $\lambda \neq \gamma$. If $\lambda > \gamma$, the solutions is as follows:

$$\pm a_3^{\frac{1}{2}} (\xi - \xi_0) = 2\sqrt{a_3^{\frac{1}{3}} U - \lambda} + \frac{\lambda}{\sqrt{\gamma - \lambda}} \arctan \frac{\sqrt{a_3 U - \gamma}}{\sqrt{\gamma - \lambda}}. \tag{29}$$

Case2: $\Delta = 0$, $D_1 = 0$, then we have $F(U) = (U - \lambda)^3$, the solution is as follows:

$$\pm a_3^{\frac{1}{2}} (\xi - \xi_0) = 2\sqrt{a_3^{\frac{1}{3}} U - \lambda} - \frac{\lambda}{\sqrt{a_3^{\frac{1}{3}} U - \gamma}}. \tag{30}$$

Case3: $\Delta > 0$, $D_1 < 0$, then $F(U) = (U - \lambda)(U - \gamma)(U - \kappa)$, we suppose that $\lambda < \gamma < \kappa$. It is easy to see the corresponding integral can be expressed by the second kind of elliptic integrals.

Case4:$\Delta < 0$, then we have $F(U) = (U - \lambda)(U^2 + pU + q), p^2 - 4q < 0$. Then the corresponding integral can be expressed by the second kind of elliptic integrals.

When $\beta = 2$, then Eq.(14) becomes

$$\int \frac{v dv}{\sqrt{a_4 v^4 + a_3 v^3 + a_2 v^2 + a_0}} = \pm(\xi - \xi_0), \tag{31}$$

where $\alpha_4 = \frac{d_2}{4}$, $a_3 = \frac{d_1}{3}$, $a_2 = \frac{d_0}{2}$, a_0 is an arbitrary constant. We take transformations as follows:

$$U = a_4^{\frac{1}{4}} (v + \frac{a_3}{4a_4}). \tag{32}$$

Then Eq.(31) becomes

$$\int\frac{(U+c)dv}{\sqrt{U^4+pU^2+qU+r}}=\pm a_4^{\frac{1}{2}}(\xi-\xi_0), \tag{33}$$

where $c=4a_4^{-\frac{3}{4}}a_3$, $p=\frac{a_2}{\sqrt{a_4}}$, $q=a_4(\frac{a_3^3}{8a_4^2}-\frac{a_2a_3}{2a_4}+a_1)$, $\gamma=a_0-\frac{a_1a_3}{4a_4}+\frac{a_2a_3^2}{16a_4^2}-\frac{3a_3^4}{256a_4^3}$

According to the classification to ODE $(U')^2=U^4+pU^2+qU+\gamma$, we can give all traveling wave solutions to Eq.(30). We omit them for simplicity.

When $\beta=3$, then Eq.(14) becomes

$$\int\frac{v^{\frac{3}{2}}dv}{\sqrt{a_5v^5+a_3v^4+a_1v^3+a_0}}=\pm(\xi-\xi_0), \tag{34}$$

where $\alpha_5=\frac{d_2}{5}$,$\alpha_4=\frac{d_1}{4}$, $\alpha_3=\frac{d_0}{3}$, a_0 is an arbitrary constant. We take transformation as following:

$$U=a_5^{\frac{1}{5}}\nu+\frac{4}{5}a_4a_5^{-\frac{4}{5}}. \tag{35}$$

Then Eq.(34) becomes

$$\int\frac{(U+c)^{\frac{3}{2}}dv}{\sqrt{U^5+pU^3+qU^2+rU+s}}=\pm a_5^{\frac{2}{5}}(\xi-\xi_0), \tag{36}$$

where

$$p=\frac{16}{5}a_4^2a_5^{-\frac{8}{5}}+a_3a_5^{-\frac{3}{5}}, q=-\frac{32}{25}a_4^3a_5^{-\frac{2}{5}}+(a_2-3a_3)a_5^{-\frac{2}{5}},$$
$$r=\frac{256}{128}a_4(a_5^{-4}-a_5^{-\frac{16}{5}})+\frac{48}{25}a_3a_4a_5^{-\frac{11}{5}}-\frac{8}{5}a_2a_4a_5^{-\frac{6}{5}}+a_1a_5^{-\frac{1}{5}},$$
$$s=\frac{256}{3125}a_4^5a_5^{-4}-\frac{64}{125}a_3a_4^3a_5^{-3}+\frac{16}{25}a_2a_4a_5^{-1}+a_0,$$

According to the classification to ODE $(U')^2=U^5+pU^3+qU^2+rU+s$, we can give all traveling wave solutions to Eq.(34). We omit them for simplicity.

According to the complete discrimination system for polynomial, We obtained that all different traveling wave solutions to the $\alpha-\beta$ Family rely on β .

4 Conclusions

In summary, although the classifications of all single traveling wave solutions to nonlinear partial differential equations are a rather difficult problem. But there are a lot of nonlinear differential equations who's all traveling wave solutions can be obtained using direct integral method and complete discrimination system for polynomial. On the other hand, if a nonlinear differential equation that's reduced ODE can't be obtained by simple integral method, then we need to find more powerful tricks and methods to do this thing. We use symmetry group to reduce the order of ODE, and furthermore reduce the equation to an integral ODE. Using complete discrimination system for polynomial, we obtain the classifications of all single traveling wave solutions to some nonlinear partial differential equations. The methods and tricks used here can be expected to develop to solve more complex and more extensive equations.

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