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Well-posedness for the Viscous Weakly Dispersive Degasperis-Procesi Equation

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Abstract: In this paper, We would like to consider the viscous dispersive Degasperis-Procesi equation.we apply Kato's theorem , study the local well-posedness problem of the Cauchy problem for the viscous dispersive Degasperis-Procesi equation.We are interested in the effect of the weakly dispersive term to the viscous Degasperis-Procesi equation.

Keywords: well-posedness; viscous dispersive Degasperis-Procesi equation; Kato's theory

1 Introduction

Recently, Holm and Staley [1] introduced the b-family PDEs that described the balance between convection and stretching for small viscosity in the dynamics of one dimensional nonlinear wave in fluids

$$m_t + um_x + bu_x m = \varepsilon m_{xx} \tag{1.1}$$

Here $u = g * m$, denotes $u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy$. The convolution relates velocity u to momentum density m by integration against the kernel $g(x)$.

When Eq.(1.1) is restricted to the peakon case $g(x) = e^{-|x|/\alpha}$ with length scale α and $m = u - \alpha^2 u_{xx}$, it may be expressed solely in terms of the velocity $u(x,t)$ as (see [1]).

$$u_t - \alpha^2 u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + (b+1)uu_x = \alpha^2(bu_x u_{xx} + uu_{xxx}), \tag{1.2}$$

where b, α are arbitrary real constants. Holm and Staley studied the effects of the balance parameter b and kernel $g(x)$ of solitary wave structures and investigated their interactions analytically for $\varepsilon = 0$ and numerically for small viscosity $\varepsilon \neq 0$, of [1].

With $\varepsilon = 0$ in Eq.(1.2), it becomes the usual b-family equation

$$u_t - \alpha^2 u_{xxt} + (b+1)uu_x = \alpha^2(bu_x u_{xx} + uu_{xxx}). \tag{1.3}$$

The Eq.(1.3) can be derived as the as the family of asymptotically equivalent shallow water wave equations that emerge at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation, of [2,3]. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated, of [2,3].

If $\alpha = 0$, $b = 2$ and $\varepsilon = 0$, then Eq.(1.2) becomes the well-known Korteweg-de Vries equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity, of [4-5]. For $\alpha = 1$, $b = 2$ and $\varepsilon = 0$, Eq. (1.2) becomes the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, the Cauchy problems of the Camassa-Holm equation have been studied extensively. It has been shown that this equation is locally well-posed [6-10] for the initial data $u_0 \in H^s(R)$ with $s > \frac{3}{2}$. For $\alpha = 1$, $b = 2$ and $\varepsilon \neq 0$ in Eq.(1.2), it becomes

$$u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

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which is the one-dimensional version of the three dimensional Navier-Stokes-alpha model for turbulence [11,12], we call Eq. (1.4) the viscous Camassa-Holm equation.

If $\alpha = 1$, $b = 3$ and $\varepsilon = 0$ in Eq. (1.2), then we find the Degasperis-Procesi equation. The Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water waves dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation. Recently, the problem of global well-posednes for the Cauchy problem of the viscous Degasperis-Procesi equation is studied[13].

$$u_t - u_{xxt} - (u_{xx} - u_{xxxx}) + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.5}$$

We would like to consider the viscous dispersive Degasperis-Procesi equation

$$u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + 4uu_x + L(u) = 3u_x u_{xx} + uu_{xxx} \tag{1.6}$$

where $L(u)$ is a dispersive item, L can be a differential operator or a quasi-differential operator according to different physical situations. We are interested in the effect of the weakly dispersive term to the viscous Degasperis-Procesi equation.

In this letter, we are interested in well-posedness problem for the following equation

$$u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + 4uu_x + \gamma(u - u_{xx})_x = 3u_x u_{xx} + uu_{xxx} \tag{1.7}$$

where $\varepsilon(u_{xx} - u_{xxxx})$ is viscous item, $L(u) = \gamma(u - u_{xx})_x$ is the weakly dispersive item and $\gamma > 0$ is constant. The optimal control problem for the Eq.(1.7) has been studied in [14].

Notation we denote by " $*$ " the convolution. We write $\hat{u}$ as the Fourier transform of u and also use $(,)$ to present the standard inner product in L^2, for $1 \leq p \leq \infty$. The norm in the Lebesgue space L^p will be written $\|.\|_{L^p}$, while $\|.\|_s$, $s \geq 0$ will stand for the norm in the classical Sobolev space $H^s(R)$. If A is unbounded operator, $D(A)$ is defined as the domain of the operator A, $[A, B]$ denotes the commuter of the linear operators A, B. $\|.\|_s$ is defined as the norm in Banach space X.

2 Local well-posedness

In this section, we apply Kato's theory to establish local well-posedness for the Cauchy problem of (1.7). For convenience, we state here Kato's theorem in a form suitable for our purpose. Consider the abstract quasi-linear evolution equation:

$$\begin{cases} \frac{dv}{dt} + A(v)\,v = f(v)\,,\ t \geq 0, \\ \qquad v(0) = v_0 \end{cases} \tag{2.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ denote the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. The linear operator A belongs to $G(X, 1, \beta)$, where β is a real number, $-A$ generates is a C_0-semigroup such that $\left\|e^{-sA}\right\|_{L(X)} \leq e^{\beta s}$. We make the following assumptions, where μ_1, μ_2, μ_3 and μ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$.

(i) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\|(A(y) - A(z))\,w\|_X \leq \mu_1 \|y - z\|_X \|w\|_Y \qquad y, z, w \in Y$$

and $A(y) \in G(X, 1, \beta)$ (i.e. $A(Y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)\,Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y) - B(z))\,w\|_X \leq \mu_2 \|y - z\|_Y \|w\|_X \qquad y, z \in Y, w \in X$$

.

(iii) $f : Y \to Y$ extends to a map from Y into Y, is bounded on bounded sets in Y, and satisfies

$$\|f(y) - f(z)\|_Y \leq \mu_3 \|y - z\|_Y \qquad y, z \in Y,$$

$$\|f(y) - f(z)\|_X \leq \mu_4 \|y - z\|_X \qquad y, z \in Y.$$

Theorem 2.1 *(Kato, [15]). Assume that (i), (ii), and (iii) hold. Given* $v_0 \in Y$*, there is a maximal* $T > 0$ *depending only on* $\|v_0\|_Y$*, and a unique solution* v *to (2.1) such that* $v = v(\cdot, v_0) \in C([0,T);Y) \cap C^1([0,T);X)$ *. Moreover, the map* $v_0 \to v(\cdot, v_0)$ *is a continuous map from* Y *to* $C([0,T);Y) \cap C^1([0,T);X)$*.*

With $m = u - u_{xx}$, we consider the Cauchy problem of Eq.(1.7)

$$\begin{cases} m_t - \varepsilon m_{xx} + 3u_x m + u m_x + \gamma m_x = 0, & t > 0,\ x \in R, \\ m(0,x) = u_0(x) - u_{0,xx}(x), & x \in R. \end{cases} \tag{2.2}$$

Note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in R$, then $\left(1 - \partial_x^2\right)^{-1} f = p * f$ for all $f \in L^2(R)$, and $p * m = u$. Using this identity, we can rewrite (2.2) as

$$\begin{cases} u_t + uu_x - \varepsilon u_{xx} + \gamma u_x = -\partial_x p * \left(\frac{3}{2}u^2\right), & t > 0, x \in \mathrm{R}\ , \\ u(0,x) = u_0(x)\ , \quad x \in R. \end{cases} \tag{2.3}$$

Definition 2.2 *If* $u \in C([0,t), H^s(R)) \cap C^1([0,t), H^{s-1}(R))$ *with* $s > \frac{3}{2}$ *satisfies (2.3), then u is called a strong solution to (2.3). If u is a strong solution on* $[0,T)$ *for every* $T > 0$*, then it is called global strong solution to (2.3).*

The local well-posedness of the Cauchy problem of (2.3) with initial data $u_0 \in H^s(R),\ s > \frac{3}{2}$ can be obtained by applying Kato's theorem [19]. More precisely, we have the following well-posedness result.

Theorem 2.3 *Given* $u_0 \in H^s(R), (s > \frac{3}{2})$*, there exists a maximal value* $T = T(\alpha, c_0, \gamma, \varepsilon, u_0) > 0$*, and a unique solution u to (2.3), such that* $u = u(\cdot, u_0) \in C([0,T);H^s) \cap C^1\left([0,T);H^{s-1}(R)\right)$*. Moreover, the solution depends continuously on the initial data, i.e., the mapping* $u_0 \to u(\cdot, u_0) : H^s \to C([0,T);H^s) \cap C^1\left([0,T);H^{s-1}(R)\right)$ *is continuous.*

To prove this theorem, we apply Theorem 2.1 with $A(u) = (u + \gamma - \varepsilon u_x)\partial_x$, $f(u) = -\partial_x\left(1 - \partial_x^2\right)^{-1}\left(\frac{3}{2}u^2\right)$, $Y = H^s, X = H^{s-1}, Q = \wedge = \left(1 - \partial_x^2\right)^{\frac{1}{2}}$, obviously, Q is an isomorphism of H^s onto H^{s-1}. Thus, in order to derive Theorem 2.3 from Theorem 2.1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions (i)-(iii). We introduce some useful lemmas.

Lemma 2.4 *([15]) Let* s, t *be real numbers such that* $-s < t \le s$*. Then*

$$\|fg\|_t \le c\|f\|_s \|g\|_t, \quad if \quad s > \frac{1}{2},$$

$$\|fg\|_{s+t-\frac{1}{2}} \le c\|f\|_s \|g\|_t, \quad if \quad s < \frac{1}{2},$$

where c is a positive constant depending on s and t.

Lemma 2.5 *([16]) Let* $f \in H^s, s > \frac{3}{2}$*. Then*

$$\left\|\wedge^{-\gamma}\left[\wedge^{\gamma+t+1}, M_f\right]\wedge^{-t}\right\|_{L(L^2)} \le c\|f\|_s \qquad |\gamma|, |t| \le s - 1,$$

where M_f *is the operator of multiplication by* f*, and c is a constant depending only on* γ *and t.*

Lemma 2.6 *([17]) Let* X *and* Y *be two Banach spaces and* Y *be continuously and densely embedded in* X*. Let* $-A$ *be the infinitesimal generator of the* C_0*-semigroup* $T(t)$ *on* X *and let* S *be an isomorphism from* Y *onto* X*. Then* Y *is* $-A$*-admissible if and only if* $-A_1 = -SAS^{-1}$ *is the infinitesimal generator of the* C_0*-semigroup* $T_1(t) = ST(t)S^{-1}$ *on* X*. Moreover, if* Y *is* $-A$*-admissible , then the part of* $-A$ *in* Y *is the infinitesimal generator of the restriction of* $T(t)$ *to* Y*.*

Lemma 2.7 *The operator* $A(u) = (u + \gamma - \varepsilon u_x)\partial_x$*, with* $u \in H^s, s > \frac{3}{2}$*, belongs to* $G\left(L^2, 1, \beta\right)$*.*

Proof. Since L^2 is a Hilbert space, we have $A\left(u\right) \in G\left(L^2, 1, \beta\right)$ for some real number β if and only if the following conditions hold:

(a) $\left(A\left(u\right)m, m\right)_0 \geq -\beta\left\|m\right\|_0^2$

(b) The range of $A + \lambda$ is all of X, for some (or all) $\lambda > \beta$.

We first prove (a). Since $u \in H^s, s > \frac{3}{2}$, u, u_x, u_{xx} belong to L^∞. Note that $\left\|u_x\right\|_{L^\infty} \leq \left\|u\right\|_s$ and $\left\|u_{xx}\right\|_{L^\infty} \leq \left\|u\right\|_s$. Thus

$$\begin{aligned}(A(u)m, m)_0 &= (u\partial_x m, m)_0 + \gamma(\partial_x m, m)_0 - \varepsilon(u_x\partial_x m, m)_0\\ &= -\tfrac{1}{2}(u_x m, m)_0 - \tfrac{\varepsilon}{2}(u_{xx}m, m)_0\\ &\leq (\tfrac{1}{2}\left\|u_x\right\|_{L^\infty} + \tfrac{\varepsilon}{2}\left\|u_{xx}\right\|_{L^\infty})\left\|m\right\|_0^2\\ &\leq c\left\|u\right\|_s\left\|m\right\|_0^2\end{aligned} \quad .$$

Setting $\beta = c\left\|u\right\|_s$, we obtain $\left(A\left(u\right)m, m\right)_0 \geq -\beta\left\|m\right\|_0^2$.

Next, we prove (b). Because $A(u)$ is a closed operator and satisfies (a), $(\lambda I + A)$ has closed range in L^2 for all $\lambda > \beta$. Therefore, it suffices to show that $(\lambda I + A)$ has dense range in L^2 for all $\lambda > \beta$.

Give $u \in H^s, s > \frac{3}{2}$, and $m \in L^2$, we have the generalized Leibnitz formula

$$\partial_x(u + \gamma - \varepsilon u_x)m = (u_x - \varepsilon u_{xx})m + (u + \gamma - \varepsilon u_x)\partial_x m.$$

Since $u_x,\ u_{xx} \in L^\infty$, we have

$$\begin{aligned}D(A) = D((u + \gamma - \varepsilon u_x)\partial_x) &= \left\{m \in L^2,\ (u + \gamma - \varepsilon u_x)\partial_x m \in L^2\right\}\\ &= \left\{z \in L^2,\ -\partial_x(u + \gamma - \varepsilon u_x)z \in L^2\right\}\\ &= D(((u + \gamma - \varepsilon u_x)\partial_x)^*) = D(A^*)\end{aligned}$$

Assume that the range of $(A + \lambda I)$ is not all of L^2. Then there exists $z \in L^2, z \neq 0$ such that $\left(\left(\lambda I + A\right)m, z\right)_0 = 0$ for all $m \in D(A)$. Since $H^1 \subset D\left(A\right)$, $D(A)$ is dense in L^2. Hence it follows that $z \in D\left(A^*\right)$ and $\lambda z + A^* = 0$ in L^2. Since $D\left(A\right) = D\left(A^*\right)$, multiplying by z and integrating by parts, we obtain

$$0 = \left(\left(\lambda I + A^*\right)z, z\right)_0 = (\lambda z, z) + (z, Az) \geq (\lambda - \beta)\left\|z_0\right\|_0^2, \quad \forall\lambda > \beta$$

and thus $z = 0$, which contradicts our assumption $z \neq 0$.This completes the proof of Lemma 2.7. ■

Lemma 2.8 *The operator* $A(u) = (u + \gamma - \varepsilon u_x)\partial_x$, *with* $u \in H^s, s > \frac{3}{2}$, *belongs to* $G\left(H^{s-1}, 1, \beta\right)$.

Proof. Since H^{s-1} is a Hilbert space, we have $A\left(u\right) \in G\left(H^{s-1}, 1, \beta\right)$ for some real number β if and only if the following conditions hold:

(a) $\left(A\left(u\right)m, m\right)_{s-1} \geq -\beta\left\|m\right\|_{s-1}^2$

(b) $-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}, for some (or all)$\lambda > \beta$.

We first prove (a). Since $u \in H^s, s > \frac{3}{2}$, u, u_x and u_{xx} belong to L^∞. Note that $\left\|u_x\right\|_{L^\infty} \leq \left\|u\right\|_s$, $\left\|u_{xx}\right\|_{L^\infty} \leq \left\|u\right\|_s$ and $\left(\wedge^{s-1}\gamma\partial_x m, \wedge^{s-1}m\right)_0 = 0$, then we have

$$\begin{aligned}\left(A\left(u\right)m, m\right)_{s-1} &= \left(\wedge^{s-1}\left(u\partial_x m\right), \wedge^{s-1}m\right)_0 + (\Lambda^{s-1}(-\varepsilon u_x\partial_x m), \Lambda^{s-1}m)_0\\ &= \left(\left[\wedge^{s-1}, u\right]\partial_x m, \wedge^{s-1}m\right)_0 - \tfrac{1}{2}\left(u_x \wedge^{s-1} m, \wedge^{s-1}m\right)_0\\ &\quad +(\left[A^{s-1}, -\varepsilon u_x\right]\partial_x m, A^{s-1}m)_0 - \tfrac{1}{2}(-\varepsilon u_{xx}A^{s-1}m, A^{s-1}m)_0\\ &\leq \left\|\left[\wedge^{s-1}, u\right]\wedge^{2-s}\right\|_{L(L^2)}\left\|\wedge^{s-1}m\right\|_0^2 + \left\|u_x\right\|_{L^\infty}\left\|\wedge^{s-1}m\right\|_0^2\\ &\quad + \left\|\left[\wedge^{s-1}, -\varepsilon u_x\right]\wedge^{2-s}\right\|_{L(L^2)}\left\|\wedge^{s-1}m\right\|_0^2 + \left\|u_{xx}\right\|_{L^\infty}\left\|\wedge^{s-1}m\right\|_0^2\\ &\leq c\left\|u\right\|_s\left\|m\right\|_{s-1}^2\end{aligned}$$

By Lemma 2.5 with $\gamma = 0, t = s - 2$. Setting $\beta = c\left\|u\right\|_s$, we obtain $\left(A\left(u\right)m, m\right)_{s-1} \geq -\beta\left\|m\right\|_{s-1}^2$ as claimed.

Next, we prove (b). Let $S = \Lambda^{s-1}$, and note that S is an isomorphism of H^{s-1} onto L^2 and that H^{s-1} is continuously and densely embedded in L^2 since $s > \frac{3}{2}$. Define

$$A_1\left(u\right) = SA\left(u\right)S^{-1} = \wedge^{s-1}A\left(u\right)\wedge^{1-s}, \qquad B_1\left(u\right) = A_1\left(u\right) - A\left(u\right)$$

Let $m \in L^2$ and $u \in H^s, s > \frac{3}{2}$. Then we have

$$\begin{aligned}\|B_1(u)m\|_0 &= \left\|\left[\wedge^{s-1}, u\partial_x\right] \wedge^{1-s} m\right\|_0 + \left\|\left[\Lambda^{s-1}, -\varepsilon u_x\partial_x\right] \wedge^{1-s} m\right\|_0 \\ &\le \left\|\left[\wedge^{s-1}, u\right] \wedge^{2-s}\right\|_{L(L^2)} \left\|\wedge^{-1}\partial_x m\right\|_0 + \left\|\left[\wedge^{s-1}, -\varepsilon u_x\right] \wedge^{2-s}\right\|_{L(L^2)} \left\|\wedge^{-1}\partial_x m\right\|_0 \\ &\le c\,\|u\|_s \|m\|_0 .\end{aligned}$$

By Lemma 2.5 with $\gamma = 0, t = s-2$. Hence $B_1(u) \in L\left(L^2\right)$.

Note that $A_1(u) = A(u) + B_1(u)$ and $A(u) \in G\left(L^2, 1, \beta\right)$, by a perturbation theorem for semigroups, we obtain $A_1(u) \in G\left(L^2, 1, \beta\right)$. Applying Lemma 2.6 with $Y = H^{s-1}$, $X = L^2$, and $S = \Lambda^{s-1}$, we conclude that H^{s-1} is A-admissible. Hence $-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}. This completes the proof of Lemma 2.8. ■

Lemma 2.9 *The operator* $A(u) = (u + \gamma - \varepsilon u_x)\partial_x$, *with* $u \in H^s, s > \frac{3}{2}$, *then* $A(u) \in L\left(H^s, H^{s-1}\right)$. *Moreover,*

$$\|(A(u) - A(z))w\|_{s-1} \le \mu_1 \|u - z\|_{s-1} \|w\|_s, \qquad u, z, w \in H^s \tag{2.4}$$

Proof. Let $u, z, w \in H^s,\ s > \frac{3}{2}$. Note that H^{s-1} is a Banach algebra. Then we have

$$\begin{aligned}\|(A(u) - A(z))w\|_{s-1} &\le c\,\|u - z\|_{s-1} \|(u_x - z_x)\|_{s-1} \|\partial_x w\|_{s-1} \\ &\le c\,\|u - z\|_{s-1} \|u - z\|_s \|w\|_s \\ &\le \mu_1 \|u - z\|_{s-1} \|w\|_s .\end{aligned}$$

This let $\mu_1 = c\,\|(u - z)\|_s$. Taking $z = 0$ in the above inequality, we obtain $A(u) \in L\left(H^s, H^{s-1}\right)$. This completes the proof of Lemma 2.9. ■

Lemma 2.10 $B(u) = \left[\Lambda^{-1}, (u + \gamma - \varepsilon u_x)\partial_x\right]\Lambda^{-1} \in L(H^{s-1})$ *for* $u \in H^s, s > \frac{3}{2}$. *Moreover*

$$\|(B(u) - B(z))w\|_{s-1} \le \mu_2 \|u - z\|_s \|w\|_{s-1}, \quad u, z \in H^s,\ w \in H^{s-1}$$

Proof. We have

$$\begin{aligned}\|(B(u) - B(z))w\|_{s-1} &\le \left\|\Lambda^{s-1}\left[\wedge^1, (u - z - \varepsilon u_x + \varepsilon z_x)\partial_x\right] \wedge^{-1} w\right\|_0 \\ &\le \left\|\Lambda^{s-1}\left[\Lambda, (u - z - \varepsilon(u_x - z_x))\right]\Lambda^{1-s}\right\|_{L(L^2)} \left\|\Lambda^{s-2}\partial_x w\right\|_0 \\ &\le \mu_2 \|y - z\|_s \|w\|_{s-1},\end{aligned}$$

where we applied Lemma 2.5 with $\gamma = 1 - s, t = s - 1$. Taking $z = 0$ in the above inequality, we obtain$B(u) \in L\left(H^{s-1}\right)$. This completes the proof of Lemma 2.10. ■

Lemma 2.11 *Let* $f(u) = -\partial_x\left(1 - \partial_x^2\right)^{-1}\left(\frac{3}{2}u^2\right)$, *then it is bounded on bounded sets in* H^s, *and satisfies*

(a) $\|f(y) - f(z)\|_s \le \mu_3 \|y - z\|_s, \qquad y, z \in H^s,$

(b) $\|f(y) - f(z)\|_{s-1} \le \mu_4 \|y - z\|_{s-1}, \qquad y, z \in H^s.$

Proof. Let $y, z \in H^s, s > \frac{3}{2}$, and note that H^{s-1} is a Banach algebra. Then we have

$$\begin{aligned}\|f(y) - f(z)\|_s &= \left\|-\partial_x\left(1 - \partial_x^2\right)^{-1}\left(\frac{3}{2}\left(y^2 - z^2\right)\right)\right\|_s \\ &\le C\,\|(y - z)(y + z)\|_{s-1} \\ &\le C\,\|y - z\|_s \|y + z\|_s \\ &\le C\left(\|y\|_s + \|z\|_s\right)\|y - z\|_s .\end{aligned}$$

This proves (a). Taking $z = 0$ in the above inequality, we obtain that f is bounded on bounded sets in H^s.

Next, we prove (b). Let $y, z \in H^s, s > \frac{3}{2}$, and note that H^{s-1} is a Banach algebra. Then we have

$$\begin{aligned}\|f(y) - f(z)\|_{s-1} &= \left\|-\partial_x\left(1 - \partial_x^2\right)^{-1}\left(\frac{3}{2}\left(y^2 - z^2\right)\right)\right\|_{s-1} \\ &\le C\,\|(y - z)(y + z)\|_{s-2} \\ &\le C\,\|y - z\|_{s-1} \|y + z\|_{s-1} \\ &\le C\left(\|y\|_{s-1} + \|z\|_{s-1}\right)\|y - z\|_{s-1},\end{aligned}$$

where we applied Lemma 2.4 with $\gamma = 1 - s, t = s - 1$. This completes the proof of Lemma 2.11. ■

Proof of Theorem 2.3. The result follows by combining Theorem 2.1 and Lemmas 2.8-2.11.

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