ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.2,pp.218-222

Products of Recurrent Non-wandering Semigroups

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(Received 2 June 2009, accepted 20 September 2009)

Abstract: In the paper, the product of strongly continuous semigroups acting on infinite dimensional separable Banach space is studied. By introducing the non-wandering criterion (NWC) and the recurrent non-wandering criterion (RNWC), a sufficient condition for $T(t) \times S(t)$ being non-wandering is given. Meanwhile, the conclusion is generalized.

Keywords: non-wandering operator; non-wandering criterion; strongly continuous semigroups; product

1 Introduction

Research on hypercyclic and chaotic semigroups has been performed since the late 1960's (see [1]), with intense work for the last three decades. Discrete hypercyclic semigroups (i.e., powers of hypercyclic bounded operators) have been treated (see [2-3]). Hypercyclicity for continuous semigroups has been observed in context with first order partial differential equations, frequently connected with models from structured population dynamics, dynamics of cell-growth and mathematical epidemiology (see [4-6]). Hypercyclicity of translation semigroups is considered (see [7]). A good overview and more literature is found (see [8-9]).Recently much attention has been paid to the non-wandering operator and non-wandering semigroups. Jiangbo Zhou, etc discussed the hereditayily hypercyclic decomposition of non-wandering operators in infinite dimensional Frechet space (see [10]); Xun Liu, etc discussed non-wandering semigroup (see [11]); Shaoguang Shi, etc studied the non-wandering property of semigroups (see [12]) and Lihong Ren, etc studied n-multiple non-wandering operator (see [13]). Lixin Tian, Shaoguang Shi, etc studied the invariance of nonwandering operator under small perturbation (see[14]). Lixin Tian, Minggang Wang studied Pseudo orbit tracing property of non-wandering operator (see[15]). Huan Qian studied the recurrent set and R stability of non-wandering operator (see[16]).

On the basis of the above research, we investigate products of recurrent non-wandering semigroup. This remainder of the paper is organized as follows. In Section 2, the basic notations and definitions are listed. Then in Section 3 and Section 4, the non-wandering criterion (NWC) and the recurrent non-wandering criterion (RNWC) are introduced respectively. And the products of strongly continuous semigroups is studied.

2 Basic notation and definitions

Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space on real number field or complex number field K. Let $L(X)$ be the set of all bounded linear operators over X. N, Z, Q, R and C will be referred to as the sets of positive integers, rational numbers, and the real and complex scalar fields, respectively.

Definition 1 (see [17]) *Suppose $T \in L(X)$. T is a linear chaotic operator or a linear chaotic map if it satisfies the following two conditions:*
(1) T is topologically transitive, i.e., T has a dense orbit in X;
(2) The set of periodic points $Per(T)$ for T is dense in X.

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IJNS.2009.10.15/270

Definition 2 (see [12]) *Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space. Suppose $\{T(t)\} \subset L(X)$ $(t \in J)$. Then $\{T(t)\}$ is called a non-wandering semigroup if it satisfies:*
(1) There exists a closed subspace $E \subseteq X$, which has hyperbolic structure:$E = E^u \oplus E^s$, $T(t)E^u = E^u$,$T(t)E^s = E^s$,where E^u, E^s are closed subspaces. In addition, there exist constants $\tau(0 < \tau < 1)$ and $C > 0$, such that $\|T(t)^k\xi\| \geq C\tau^{-k}\|\xi\|$, for any $\xi \in E^u, k \in N$; $\|T(t)^k\eta\| \leq C\tau^{-k}\|\eta\|$, for any $\eta \in E^s, k \in N$;
(2) $Per(T(t))$ is dense in E.

Remark 1 *(1) In definition 2, $\{T(t)\}(t \in J)$ can be a strongly continuous semigroup, $C-$semigroup, etc. And if $\{T(t)\}(t \in J)$ is a non-wandering semigroup, then $\overline{Per(T(t))} \bigcap E = \varnothing$.*
(2) In this paper, we study products of strongly continuous semigroups.

3 Non-wandering criterion (NWC)

To estimate the non-wandering of strongly continuous semigropups effectively, we introduce the non-wandering criterion (NWC).

Definition 3 *(non-wandering criterion) A strongly continuous semigroup $\{T(t)|t \geq 0\}$ on X satisfies the non-wandering criterion (NWC) if*
(1) There exists a closed subspace $E \subseteq X$, which has hyperbolic structure:$E = E^u \oplus E^s$, $T(t)E^u = E^u$,$T(t)E^s = E^s$,where E^u, E^s are closed subspaces. In addition, there exist constants $\tau(0 < \tau < 1)$ and $C > 0$, such that $\|T(t)^k\xi\| \geq C\tau^{-k}\|\xi\|$, for any $\xi \in E^u, k \in N$; $\|T(t)^k\eta\| \leq C\tau^{-k}\|\eta\|$, for any $\eta \in E^s, k \in N$;
(2) For all nonempty open sets $U \subset E$ there exists some $t \geq 0$ such that $T^k(t)U \cap U \neq \emptyset (k = 1, 2, \cdots)$.

Remark 2 *Definition 3 is from Definition 2, but is different from it.*

Proposition 1 *Let $\{T(t)|t \geq 0\}$ be a strongly continuous semigroup on infinite dimensional separable Banach space X. If $\{T(t)|t \geq 0\}$ satisfies the non-wandering criterion, then $\{T(t)|t \geq 0\}$ is a non-wandering semigroup.*

Proof. Compare definition 3 and definition2, we only proof T(t) satisfy (2) in definition 2 by using (2) in definition 3. For $\forall x_0 \in E, \varepsilon > 0, U(x_0, \varepsilon)$,from the non-wandering criterion, we have $T^k(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon)(k = 1, 2, \cdots)$. Particularly, we have

$$\begin{cases} k = 1, T(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon) \neq \emptyset, \exists y_0 \in T(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon) \\ k = 2, T^2(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon) \neq \emptyset, \exists y_1 \in T^2(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon), \| T(t)y_0 - y_1 \| < \varepsilon \\ \ldots\ldots \\ k = N, T^N(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon) \neq \emptyset, \exists y_{n-1} \in T^N(t)U(x_0, \varepsilon) \cap U(x_0, \varepsilon), \| T(t)y_{n-2} - y_{n-1} \| < \varepsilon \\ \ldots\ldots \end{cases}$$

Then we have

$$\begin{cases} \| T(t)y_0 - y_1 \| < \varepsilon \\ \| T^2(t)y_0 - y_2 \| \leq (\| T(t) \| + 1)\varepsilon \\ \ldots\ldots \\ \| T^N(t)y_0 - y_n \| \leq (\| T(t) \|^{N-1} + \| T(t) \|^{N-2} + \cdots + \| T(t) \| + 1)\varepsilon \\ \ldots\ldots \end{cases}$$

In fact, since $T(t)$ is bounded, we only consider $\| T(t) \| \leq 1$. So

$$\begin{aligned} \| T^N(t)y_n - y_n \| &\leq \| T^N(t)(y_n - y_0) \| + \| T^N(t)y_0 - y_n \| \\ &\leq \| T(t) \|^N \varepsilon + (\| T(t) \|^{N-1} + \| T(t) \|^{N-2} + \cdots + \| T(t) \| + 1)\varepsilon < (N + 1)\varepsilon. \end{aligned}$$

Next we set $y_n = p$, then $\| T^N(t)p - p \| < (N + 1)\varepsilon$. And choosing different ε we can get $\{t_n\}, \{p_m\}$, since X is an infinite dimensional separable Banach space, then there exists p^* such that $p_m \to p^*$. And on the other hand we have $\| T^N(t_n)p^* - T^N(t_m)p^* \| \leq \| T^N(t_n)p^* - p^* \| + \| T^N(t_m)p^* - p^* \| < M(N)\varepsilon$ ($M(N)$ is bounded). For a given N, since ε is enough small and $\{T(t)|t \geq 0\}$ is a strongly continuous semigroup, so there exists t^* such that $t_n \to t^*$, then we have $T^N(t^*)p^* = p^*$, p^* is a periodic point, $\overline{Per(T(t))} = E$, therefore $\{T(t)|t \geq 0\}$ is a non-wandering semigroup. ■

4 Recurrent non-wandering criterion (RNWC)

We know that pairs of strongly continuous semigroups $S(t)$ and $T(t)$ such that both satisfy the non-wandering criterion (NWC) but the product $T(t) \times S(t)$ is not non-wandering. So we define the recurrent non-wandering criterion (RNWC) as follows:

Definition 4 *(recurrent non-wandering criterion) A strongly continuous semigroup $\{T(t)|t \geq 0\}$ on X satisfies the non-wandering criterion (NWC) if*
(1) There exists a closed subspace $E \subseteq X$, which has hyperbolic structure:$E = E^u \oplus E^s$, $T(t)E^u = E^u$,$T(t)E^s = E^s$,where E^u, E^s are closed subspaces. In addition, there exist constants $\tau(0 < \tau < 1)$ and $C > 0$, such that $\|T(t)^k\xi\| \geq C\tau^{-k}\|\xi\|$, for any $\xi \in E^u, k \in N$; $\|T(t)^k\eta\| \leq C\tau^{-k}\|\eta\|$, for any $\eta \in E^s, k \in N$;
(2) For all nonempty open sets $U \subset E$ there exists some $L \geq 0$ such that each interval $[t, t+L)$ contains an s with $T^k(s)U \cap U \neq \emptyset (k = 1, 2, \cdots)$.

Proposition 2 *Let $\{T(t)|t \geq 0\}$ be a strongly continuous semigroup on infinite dimensional separable Banach space X . If $\{T(t)|t \geq 0\}$ satisfies the recurrent non-wandering criterion, then $\{T(t)|t \geq 0\}$ satisfies the non-wandering criterion.*

Proof. From the definition 3 and 4, we can get the conclusion. ■

5 Non-wandering of products

The following lemma is critical for our purpose. So we introduce it firstly.

Lemma 3 *Let $\{T(t)|t \geq 0\}$ be a strongly continuous semigroup on X satisfying the non-wandering criterion. Then for any nonempty open set U , and each $L > 0$, there exists some $t \geq 0$ such that for all $s \in [t, t+L)$, we have $T^k(s)U \cap U \neq \emptyset (k = 1, 2, \cdots)$.*

Proof. Since the range of $T(L)$ is dense, there exists $y \in U$ and for $s \in [0, L)$ such that $T(s)y \in U$. Take a neighborhood M of y such that $T(s)M \subset U$. Then for $\forall y^* \in M$, we have $T(s)(y^* - y) \in U(0, \varepsilon)$, thus $T^k(s)(y^* - y) \in U(0, \varepsilon)(k = 1, 2, \cdots)$. And since $T(s)$ is a linear operator, then $T^k(s)y^* \in U(T^k(s)y, \varepsilon)$. Set $U(T^k(s)y, \varepsilon) = N$, thus $N \subset M$ (ε is enough small). Therefore for $\forall y^* \in N$, we have $T^k(s)y^* \in U(k = 1, 2, \cdots)$. Since $T(t)$ satisfies the non-wandering criterion, there exists $t \geq 0, x \in N$ such that $T^k(t)x \in N(k = 1, 2, \cdots)$. Now take $s \in [t, t+L)$, then $T^k(s)x = T^k(s-t)T^k(t)x \in N$, therefore the conclusion holds. ■

Theorem 4 *Let $T(t), S(t)(t \geq 0)$ be strongly continuous semigroups on infinite dimensional separable Banach space X and Y respectively. If $T(t)$ satisfies the recurrent non-wandering criterion and $S(t)$ satisfies the non-wandering criterion, then $T(t) \times S(t)$ satisfies the non-wandering criterion on $X \times Y$.*

Proof. Firstly, we proof that $T(t) \times S(t)$ has hyperbolic structure. In the following, we use $\|(x, y)\| = \|x\| \cdot \|y\|$ and $T(t) \times S(t)(x, y) = (T(t)x, S(t)y)$ on $X \times Y$. Since $T(t)$ satisfies the recurrent non-wandering criterion and $S(t)$ satisfies the non-wandering criterion, then we have:
For $T(t): E_1 = E_1^u \oplus E_1^s, T(t)E_1^u = E_1^u, T(t)E_1^s = E_1^s$ and $\exists C_1, \tau_1 (0 < \tau_1 < 1)$, such that $\| T^k(t)x_u \| \geq C_1\tau_1^{-k} \| x_u \|, \forall x_u \in E_1^u, k \in N$; $\| T^k(t)x_s \| \leq C_1\tau_1^k \| x_s \|, \forall x_s \in E_1^s, k \in N$.
For $S(t): E_2 = E_2^u \oplus E_2^s, S(t)E_2^u = E_2^u, S(t)E_2^s = E_2^s$ and $\exists C_2, \tau_2 (0 < \tau_2 < 1)$, such that $\| S^k(t)y_u \| \geq C_2\tau_2^{-k} \| y_u \|, \forall y_u \in E_2^u, k \in N$; $\| S^k(t)y_s \| \leq C_2\tau_2^k \| y_s \|, \forall y_s \in E_2^s, k \in N$.
Thus for $T(t) \times S(t)$, any $x \in E_1^u, y \in E_2^u$, $T(t) \times S(t)(x, y) = (T(t)x, S(t)y)$. So $T(t) \times S(t)(E_1^u \times E_2^u) = T(t)E_1^u \times S(t)E_2^u = E_1^u \times E_2^u$, similarly $T(t) \times S(t)(E_1^s \times E_2^s) = E_1^s \times E_2^s$.

On the other hand,
$$\| (T(t) \times S(t))^k(x_u, y_u) \| = \| (T^k(t)x_u, S^k(t)y_u) \| \geq C_1\tau_1^{-k} \| x_u \| \cdot C_2\tau_2^{-k} \| y_u \| = (C_1C_2)(\tau_1\tau_2)^{-k} \| (x_u, y_u) \|, \forall x_u \in E_1^u, y_u \in E_2^u, k \in N.$$
Similarly, $\| (T(t) \times S(t))^k(x_s, y_s) \| \leq (C_1C_2)(\tau_1\tau_2)^k \| (x_s, y_s) \|, \forall x_s \in E_1^s, y_s \in E_2^s, k \in N$.

From above, $T(t) \times S(t)$ has hyperbolic structure on $(E_1^u \times E_2^u) \cup (E_1^s \times E_2^s)$.
Next we will proof that $T(t) \times S(t)$ satisfies Definition 3(2). In fact, let U_x, U_y be nonempty open sets

in X, Y respectively. Since $T(t)$ satisfies the recurrent non-wandering criterion, let L be a constant such that for all $t \geq 0$, there exists s in $[t, t+L)$ such that $T^k(s)U_x \cap U_x \neq \emptyset (k = 1, 2, \cdots)$. According to Lemma 3, there exists some $t \geq 0$ such that $S^k(s)U_y \cap U_y \neq \emptyset (k = 1, 2, \cdots)$ for each $s \in [t, t+L)$. In particular, the interval $[t, t+L)$ contains s with $T^k(s)U_x \cap U_x \neq \emptyset (k = 1, 2, \cdots)$. This s satisfies finally $(T(t) \times S(t))^k (U_x \times U_y) \cap (U_x \times U_y) \neq \emptyset (k = 1, 2, \cdots)$. Therefore the conclusion holds. ∎

Corollary 5 *Let $T(t), S(t), R(t) (t \geq 0)$ be strongly continuous semigroups on infinite dimensional separable Banach space X , Y and Z . If $T(t), S(t)$ satisfy the recurrent non-wandering criterion and $R(t)$ satisfies the non-wandering criterion, then $T(t) \times S(t) \times R(t)$ satisfies the non-wandering criterion on $X \times Y \times Z$.*

Proof. From theorem 4, $S(t) \times R(t)$ satisfies the non-wandering criterion on $Y \times Z$, and $T(t)$ satisfies the recurrent non-wandering criterion, then $T(t) \times S(t) \times R(t)$ satisfies the non-wandering criterion on $X \times Y \times Z$. ∎

Acknowledgements

Research was supported by the National Nature Science Foundation of China (No.10771088) and Nature Science Foundation of Jiangsu (No.2007098) and Outstanding Personnel Program in Six Fields of Jiangsu(No. 6-A-029).

References

[1] Rolewicz, S: On orbits of elements. *Studia Math*. 32:17-22 (1969).

[2] Godefroy, G. and J. H. Shapiro: Operators with dense, invariant cyclic vector manifolds. *J. Funct. Anal*. 98:229-269 (1991).

[3] Kitai, C: "Invariant Closed Sets for Linear Operators". *Ph.D. Thesis,University of Toronto*. 1982.

[4] ABrunovsky, P: Notes on chaos in the cell population partial differential equation. *Nonlinear Anal*. 7:167-176 (1983).

[5] Lasota, A: Stable and chaotic solutions of a first order partial differential equation. *Nonlinear Anal*. 5:1183-1193 (1981).

[6] Webb, G. F: Periodic and chaotic behavior in structured models of cellpopulation dynamics, in "Recent Developments in Evolution Equations". *Pitman Research Notes in Math*. 324:40-49 (1995).

[7] Desch, W., W. Schappacher and G. F. Webb: Hypercyclic and chaotic semigroups of linear operators. *Ergodic Theory Dynam. Systems*. 17:793-819 (1997).

[8] Grosse-Erdmann, K. G: Recent developments in hypercyclicity. *RACSAM*. 97:273-286 (2003).

[9] Grosse-Erdmann, K. G: Universal families and hypercyclic operators. *Bull. Amer. Math. Soc*. 36:345-381 (1999).

[10] Jiangbo Zhou, Lixin Tian, Dianchen Lu:The hereditarily decomposition of non-wandering operators in infinite dimensional Frechet space. *J Jiangsu University*. 22(6):88 - 91 (2001).

[11] Xun Liu, Lixin Tian: The property of non-wandering semigroup. *J. Jiangsu University*. 23(5): 9-12 (2002).

[12] Zhong Guang-sheng, Tian Li-xin: Non-wandering property of operator semigroup[J]. *Journal of Jiangsu University (Natural Science Edition)*. 26(6):116-119 (2005).

[13] Lixin Tian, Lihong Ren: N-multiple nonwandering unilateral weighted backward shift operators and the property of direct sum operators in Banach space. *International Journal of Nonlinear Science*. 2(2):104-110 (2006).

[14] Lixin Tian, Shaoguang Shi, Lihong Ren: The invariance of nonwandering operator under small perturbation. *International Journal of Nonlinear Science*. 2:(28) (2008).

[15] Lixin Tian, Minggang Wang: Pseudo orbit tracing property of non-wandering operator. *International Journal of Nonlinear Science*. 1(3):3-7 (2007).

[16] Huan Qian: Recurrent Set and R Stability of Non-wandering Operator. *International Journal of Nonlinear Science*. 7(1):108-112 (2009).

[17] Godefroy G, Shapiro J H: Operators with dense, invariant cyclic vector manifolds. *J.Funct Anal*. 98:229-269 (1991).