

## The Attractor on a Class of Viscous Nonlinear Dispersive Wave Equations

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**Abstract:** This paper aims at presenting a proof of the attractor for a class of viscous nonlinear dispersive wave equations. In this paper, the global existence of solution to this equation in  $L^2$  under the periodical condition is studied. By using the time estimate of this equation, we get the compact and bounded absorbing set and the existence of the global attractor for the viscous nonlinear dispersive wave equation is proved.

**Keywords:** viscous; a class of viscous nonlinear dispersive wave equations; global solution; attractor;

### 1 Introduction

In [1], Degasperis and Procesi introduced the following family of third order dispersive PDE conservation laws:  $u_t + c_0 u_x + r u_{xxx} - \alpha^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x$ . where  $\alpha, r, c_i, i = 0, 1, 2, 3$  are real constants. In this paper, we will study a class of viscous nonlinear dispersive wave equations like this:

$$u_t - u_{xxt} + k u_x + \alpha u u_x = u u_{xxx} + 3 u_x u_{xx}$$

where  $\alpha > 0, k$  are constants.

With  $k = 0, \alpha = 4$ , we find the Degasperis-Procesi equation:  $u_t - u_{xxt} + 4 u u_x - 3 u_x u_{xx} - u u_{xxx} = 0$ . It is another completely integrable equation of this class. In paper [10], the dynamical behaviors of a dispersive shallow water equation with viscous Degasperis-Procesi equation were investigated by Lixin Tian and Jinling Fan.

For  $k = 1, \alpha = 1$ , it is Fornberg-Whitham equation:  $u_t - u_{xxt} + u_x + u u_x - 3 u_x u_{xx} - u u_{xxx} = 0$ . This equation is derived by B.Fornberg and G.B Whitham, where  $u$  is the fluid velocity in the  $x$  direction. The equation was used to study the qualitative behaviors of wave-breaking. It is nonintegrable. In [11], Fornberg and Whitham obtained a peaked solution of the form  $u(x, t) = A \exp \left\{ -\frac{1}{2} \left| x - \frac{4t}{3} \right| \right\}$ , where  $A$  is an arbitrary constant. In [12-13], Jiangbo Zhou and Lixin Tian constructed two types of bounded traveling wave solutions, which were called kink-like and antikink-like wave solutions.

Actually, because of viscosity and pulsation, there always exists energy dissipation while the physical fluid is moving. Global attractor is a basic concept in the study of the asymptotic behavior of solutions for the nonlinear evolution equations with various dissipation. Many papers have been devoted to the global attractor research of these equations. For example: In paper [2], the global attractor and numerical simulation of a forced weakly damped MKdV equation in  $H^2(R)$  is proved by Wenxia Chen, Lixin Tian, and Xiaoyan Deng. In [3], Lixin Tian, Ruihua Tian and Jinling Fan got the global attractor for the viscous weakly damped forced Korteweg-de Vries equation in  $H^1(R)$ . In [4-5], Lixin Tian, Wenbin Zhang and Dejun Peng proved the existence of the asymptotic smoothing and global attractor of a weakly damped, forced viscous Korteweg-de Vries equation on the real line. In two-dimensional space, the attractor for the weakly damped KdV equation in belt field was proved by Lixin Tian and Ruihua Tian [6]. In [7-8], Danping Ding and Lixin Tian investigated the dynamical behaviors of Camassa-Holm equation and got the existence of

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the global attractor of dissipative C-H equation in  $H^2(R)$ . In [9], Yiping Fu and Boling Guo studied one dimensional viscous Camassa-Holm equation with a periodic boundary condition. In [14], Anna Gao and Lixin Tian proved the existence of the global attractor of the viscous damped forced Ostrovsky equation in  $L^2(R)$ .

In this paper, we consider the viscous general shallow water wave equation in  $H^2(\Omega)$  as follows:

$$u_t - u_{xxt} + ku_x + \alpha uu_x = uu_{xxx} + 3u_x u_{xx} + \varepsilon(u - u_{xx})_{xx} \tag{1.1}$$

$$u(x, 0) = u_0(x), t > 0, x \in \Omega, \Omega = [0, L], u \in H = L^2(\Omega) \tag{1.2}$$

## 2 Main theorems

In this paper, we denote by  $(\cdot, \cdot)$  the  $L^2$  inner product and by  $\|\cdot\|$  the corresponding  $L^2$  norm,  $\|u\|_{H^m(\Omega)} = \|D^m u\|_{L^2(\Omega)}$  where  $D$  is the first-order operator. The inner product here is equivalent to the nature inner product in  $H^m(\Omega)$  when  $mes(\Omega) < +\infty$ . In the paper, we denote  $\|u\|_{L^2(\Omega)} \triangleq |u|, \|Du\|_{L^2(\Omega)} \triangleq \|u\|, \|D^m u\|_{L^2(\Omega)} \triangleq |D^m u|$ , Denote  $A = -\Delta, \Delta$  is Laplace operator,  $v = u + \xi Au$ . In the case of periodic boundary condition, Esq. (1.1) can be denoted as

$$\frac{dv}{dt} + \varepsilon Av + uv_x + 3vu_x + (\alpha - 4)uu_x + ku_x = 0 \tag{2.1}$$

$$u(x, 0) = u_0 \tag{2.2}$$

$$u(0, t) = u(L, t), u'(0, t) = u'(L, t), u''(0, t) = u''(L, t), \tag{2.3}$$

where  $A$  is a self-adjoint positive operator with compact inverse. The characteristic value of  $A$  is  $\lambda_j, 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow \infty$ , when  $j \rightarrow \infty, A\omega_j = \lambda_j \omega_j$ , where  $\omega_j$  is the corresponding characteristic vector of  $A$ .

**Theorem 1** With  $u_0 \in H^l(R), l \geq 2$ , Esq.(2.1)-(2.3) have a global solution.

**Theorem 2** With  $u_0 \in H^l(R), l \geq 2$ , let  $S(t)$  denotes the semi-group of the solution operator to Esq. (2.1)-(2.3), i.e.  $S(t) : H^2(\Omega) \rightarrow H^2(\Omega), u(t) = S(t)u_0$ . Then  $S(t)$  has absorbing set.

**Theorem 3** With  $u_0 \in H^l(R), l \geq 2$ , the semi-group of the solution operator  $S(t)$  to Esq. (2.1)-(2.3) has global attractor in  $H^2(\Omega)$ .

## 3 The proof of the theorems

In the proof of Theorem 1, we use the Galerkin procedure to prove the existence of global solution.

Let  $\{\phi_j\}_{j=1}^\infty$  be an orthonormal basis of  $H$  consisting of eigenfunctions of the operator  $A$ . The Galerkin procedure for Esq. (2.1)-(2.3) is the ordinary differential system,

$$\frac{dv_m}{dt} + \varepsilon Av_m + u_m \nabla v_m + 3v_m \nabla u_m + (\alpha - 4)u_m \nabla u_m + k \nabla u_m = 0 \tag{3.1a}$$

$$u_m(0) = a_m u(0) \tag{3.1b}$$

where  $v_m = u_m + Au_m$ . Since the nonlinear term is quadratic in  $u_m$ , then based on the classical theory of ordinary differential equations, system (3.1) has a unique solution to a short interval of time. Our purpose is to show that the solution of (3.1) remains finite for all positive times which imply that  $T_m = \infty$ .

**Lemma 1** Assume  $u_m(0) \in H^l(R), l \geq 2$ , and fix any  $\varepsilon > 0$ . Then the following identity holds for any  $t \geq 0$ :

$$\begin{aligned} & | -Ah_m(t, \cdot) |^2 + (\alpha + 1) | \nabla h_m(t, \cdot) |^2 + \alpha | h_m(t, \cdot) |^2 \\ & + 2\varepsilon \int_0^t ( | -Ah_m(\tau, \cdot) |^2 + (\alpha + 1) | \nabla h_m(\tau, \cdot) |^2 + \alpha | h_m(\tau, \cdot) |^2 ) d\tau \\ & = | -Ah_m(0, \cdot) |^2 + (\alpha + 1) | \nabla h_m(0, \cdot) |^2 + \alpha | h_m(0, \cdot) |^2 \end{aligned} \tag{3.2}$$

**Proof.** Similar to [13], From Eqs.(2.1), we have:

$$\begin{cases} \frac{d}{dt}u_m + \nabla \frac{1}{2}u_m^2 + \nabla J_m = -\varepsilon Au_m, (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ AJ_m + J_m = ku_m + \frac{\alpha-1}{2}u_m^2, (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ (h_m = G_1 * u_m) \quad u_m(0) = a_m(u_0) \end{cases} \quad (3.3)$$

where

$$\alpha h_m + Ah_m = u_m \quad (3.4)$$

Multiplying (3.3) by  $h_m + Ah_m$  and integrating over  $\mathbb{R}$ , we get

$$\begin{aligned} & \int_{\mathbb{R}} \frac{d}{dt}u_m(h_m + Ah_m)dx - \varepsilon \int_{\mathbb{R}} -Au_m(h_m + Ah_m)dx \\ &= - \int_{\mathbb{R}} u_m \nabla u_m(h_m + Ah_m)dx - \int_{\mathbb{R}} \nabla J_m(h_m + Ah_m)dx \end{aligned} \quad (3.5)$$

For the right side of this identity, using (3.4), we get

$$- \int_{\mathbb{R}} u_m \nabla u_m(h_m + Ah_m)dx - \int_{\mathbb{R}} \nabla J_m(h_m + Ah_m)dx = 0 \quad (3.6)$$

For the left side of this identity, using (3.4), we get

$$\begin{aligned} & \int_{\mathbb{R}} \frac{d}{dt}u_m(h_m + Ah_m)dx - \varepsilon \int_{\mathbb{R}} -Au_m(h_m + Ah_m)dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} [\alpha h_m^2 + (\alpha + 1)(\nabla h_m)^2 + (-Ah_m)^2] dx + \\ & \quad \varepsilon \int_{\mathbb{R}} (\alpha(\nabla h_m)^2 + (\alpha + 1)(-Ah_m)^2 + (-\nabla Ah_m)^2) dx \end{aligned} \quad (3.7)$$

Substituting (3.6) (3.7) into (3.5) yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} [\alpha h_m^2 + (\alpha + 1)(\nabla h_m)^2 + (-Ah_m)^2] dx + \\ & 2\varepsilon \int_{\mathbb{R}} (\alpha(\nabla h_m)^2 + (\alpha + 1)(-Ah_m)^2 + (-\nabla Ah_m)^2) dx = 0 \end{aligned}$$

Integrating this inequality over  $[0, t]$ , we obtain (3.2). ■

**Lemma 2** Assume  $u_m(0) \in H^l(\mathbb{R})$ ,  $l \geq 2$ , and fix any  $\varepsilon > 0$ , then the following bounds for any  $t \geq 0$ :

$$|u_m(t)| \leq \sqrt{\alpha} |u_m(0)| \leq \sqrt{\alpha} |u(0)| \quad (3.8)$$

$$\|u_m(t)\| \leq \sqrt{\frac{1}{\varepsilon}} |u_m(0)| \leq \sqrt{\frac{1}{\varepsilon}} |u(0)| \quad (3.9)$$

**Proof.** similar to [13], we can easy get (3.8) and (3.9). ■

**Proof of theorem 1:** From lemma 2, we can get

$$|u_m|^2 + \|u_m\|^2 \leq \alpha |u(0)|^2 + \frac{\alpha}{\varepsilon} |u(0)|^2 \leq M \triangleq r_1 \quad (3.10)$$

We take the inner product of (3.1a) with  $u_m$  in  $\Omega$ , we have

$$\frac{1}{2} \frac{d}{dt} (|u_m|^2 + \|u_m\|^2) + \varepsilon (\|u_m\|^2 + |Au_m|^2) = a_m \int_{\Omega} u_m u_{mx} u_{mxx} dx \quad (3.11)$$

$$\left| a_m \int_{\Omega} u_m u_{mx} u_{mxx} dx \right| \leq \frac{\varepsilon}{2} (\|u_m\|^2 + |Au_m|^2) + \frac{k_3^2}{2\varepsilon} (\|u_m\|^2 + |u_m|^2)^2$$

where  $k_3$  is constant. we get inequalities from (3.11)

$$\frac{1}{2} \frac{d}{dt} (\|u_m\|^2 + \|u_m\|^2) + \varepsilon (\|u_m\|^2 + |Au_m|^2) \leq \frac{\varepsilon}{2} (\|u_m\|^2 + |Au_m|^2) + \frac{k_3^2}{2\varepsilon} (\|u_m\|^2 + |u_m|^2)^2 \quad (3.12)$$

$$\int_t^{t+r} (\|u_m(s)\|^2 + |Au_m(s)|^2) ds \leq \frac{k_3^2/\varepsilon r_1^2 r + r_1}{\varepsilon} \triangleq r_2 \quad (3.13)$$

Now take the inner product of (3.1a) with  $Au_m$  in  $\Omega$ , we have:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|Au_m|^2 + \|u_m\|^2) + \varepsilon (|Au_m|^2 + |\nabla Au_m|^2) \\ &= a_m \alpha \int_{\Omega} u_m u_{mx} u_{mxx} dx - 3a_m \int_{\Omega} u_{mx} u_{mxx}^2 dx + a_m \int_{\Omega} u_m u_{mxxx} (-u_{mxx}) dx \end{aligned}$$

According to Agmon inequality when  $n=1$ , we have  $\|\varphi\|_{L^\infty} \leq c \|\varphi\|_{L^2(\Omega)}^{1/2} \|\varphi\|_{H^1(\Omega)}^{1/2}$ , where  $c$  is a constant which only depends on  $\Omega$ . We get:

$$\begin{aligned} & \left| a_m \alpha \int_{\Omega} u_m u_{mx} u_{mxx} dx + 3a_m \int_{\Omega} u_{mx} u_{mxx}^2 dx + a_m \int_{\Omega} u_m u_{mxxx} (-u_{mxx}) dx \right| \\ & \leq \varepsilon \lambda_1 (|Au_m|^2 + \|u_m\|^2) + c_5 \|u_m\| |Au_m| (|Au_m|^2 + \|u_m\|^2) \end{aligned}$$

where  $c_4 = \max [c_3 + 6c_2, c_1\alpha]$ ,  $c_5 = \frac{c_4^2}{4\varepsilon\lambda_1}$ . Based on Young inequality, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|Au_m|^2 + \|u_m\|^2) + \varepsilon (|Au_m|^2 + |\nabla Au_m|^2) \\ & \leq \varepsilon \lambda_1 (|Au_m|^2 + \|u_m\|^2) + c_5 \|u_m\| |Au_m| (|Au_m|^2 + \|u_m\|^2) \end{aligned} \quad (3.14)$$

From Poincare inequality  $|Au_m|^2 > \lambda_1 \|u_m\|^2$ ,  $|\nabla Au_m|^2 > \lambda_1 |Au_m|^2$  and (3.14) we get

$$\frac{d}{dt} (|Au_m|^2 + \|u_m\|^2) \leq 2c_5 \|u_m\| |Au_m| (|Au_m|^2 + \|u_m\|^2) \leq c_5 (|Au_m|^2 + \|u_m\|^2)^2 \quad (3.15)$$

Through inequality (3.13) and Gronwall's inequality, we have

$$|Au_m|^2 + \|u_m\|^2 \leq \frac{r_2}{r} \exp(c_5 r_2) \triangleq r_3, t > t_0 + r, \quad (3.16)$$

where  $r, r_2, c_5$  are nonnegative constants. Integrating (3.14) over the interval  $[t, t + r]$ , we have:

$$\varepsilon \int_t^{t+r} (|Au_m(s)|^2 + |\nabla Au_m(s)|^2) ds \leq (\varepsilon \lambda_1 r_3 + \frac{c_5 r_3^2}{2}) + r_3 \triangleq r_4. \quad (3.17)$$

Take the inner product of (3.1a) with  $A^2 u_m$  in  $\Omega$  to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|Au_m|^2 + |\nabla Au_m|^2) + \varepsilon (|A^2 u_m|^2 + |\nabla Au_m|^2) \\ &= a_m \int_{\Omega} u_m u_{mxxx} u_{mxxxx} dx - a_m (2\alpha + 1) \int_{\Omega} u_m u_{mx} u_{mxxxx} dx + \\ & (3 - \alpha) a_m \int_{\Omega} u_{mx} u_{mxx} u_{mxxxx} dx \end{aligned}$$

through young inequality, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|Au_m|^2 + |\nabla Au_m|^2) + \varepsilon (|A^2 u_m|^2 + |\nabla Au_m|^2) \\ & \leq \varepsilon \lambda_1 (|Au_m|^2 + |\nabla Au_m|^2) + c_6 \|u_m\| |Au_m| (|Au_m|^2 + |\nabla Au_m|^2) \end{aligned} \quad (3.18)$$

According to Poincare inequality and Young inequality, we have

$$\frac{d}{dt} (|Au_m|^2 + |\nabla Au_m|^2) \leq \frac{c_6}{2} (\|u_m\|^2 + |Au_m|^2) (|Au_m|^2 + |\nabla Au_m|^2)$$

From (3.13),(3.17), we get:

$$\frac{c_6}{2} \int_t^{t+r} (\|u_m(s)\|^2 + |Au_m(s)|^2) ds \leq \frac{c_6}{2} r_2, \int_t^{t+r} (|Au_m|^2 + |\nabla Au_m|^2) ds \leq \frac{r_4}{\varepsilon}.$$

Through Gronwall's inequality we have:  $|Au_m|^2 + |\nabla Au_m|^2 \leq \frac{r_4}{r\varepsilon} \exp \frac{c_6 r_2}{2} \triangleq r_5, \quad t > t_0.$  (3.19)

Integrating (3.9) over  $[t, t+r]$ , we have

$$\varepsilon \int_t^{t+r} (|A^2 u_m(s)|^2 + |\nabla Au_m(s)|^2) ds \leq (\varepsilon \lambda_1 r_5 + \frac{c_6 r_3 r_5}{2}) r + r_5 \triangleq r_6. \quad (3.20)$$

Take the inner product of (3.1a) with  $A^3 u_m$  in  $\Omega$  to obtain:

$$|\nabla Au_m|^2 + |A^2 u_m|^2 \leq \triangleq r_7. \quad (3.21)$$

Now we get  $|u_m|, \|u_m\|, |Au_m|, |\nabla Au_m|$  and  $|A^2 u_m|$  which are bounded, i.e.  $v_m, |v_m|$  and  $|Av_m|$  are bounded. Then we get  $\frac{du_m}{dt}, \frac{dv_m}{dt}$  which are also bounded. Through Aubin's compactness theorem, we conclude that there is a subsequence  $u'_m$ , so that  $u'_m \rightarrow u, v'_m \rightarrow v$ . Let us replace  $u'_m, v'_m$  by  $u_m, v_m$ . Now we prove that  $u, v$  satisfy Eq. (2.1).

$$\begin{aligned} & (v_m(t), \omega) + \varepsilon \int_{t_0}^t (v_m(s), a_m A \omega) ds + \int_{t_0}^t (u_m(s) v_{mx}(s), a_m \omega) ds + 3 \int_{t_0}^t (v_m(s) u_{mx}(s), a_m \omega) ds \\ & + (\alpha - 4) \int_{t_0}^t (u_m(s) u_{mx}(s), a_m \omega) ds + k \int_{t_0}^t (a_m \omega, u_{mx}) ds = (v_m(t_0), \omega) \end{aligned}$$

similar to [13], we know for all  $\omega \in D(A)$ , we have

$$\begin{aligned} & (v(t), \omega) + \varepsilon \int_{t_0}^t (v(s), A \omega) ds + \int_{t_0}^t (u(s) v_x(s), \omega) ds + 3 \int_{t_0}^t (v(s) u_x(s), \omega) ds \\ & + (\alpha - 4) \int_{t_0}^t (u(s) u_x(s), \omega) ds + k \int_{t_0}^t (\omega, u_x) ds = (v(t_0), \omega) \end{aligned}$$

Above all, we can conclude that the global solution to Esq. (2.1)-(2.3) exists.

**Proof of Theorem 2.** We take the inner product of (2.1) with  $u$  in  $\Omega$ .

$$\begin{aligned} & \left( \frac{dv}{dt}, u \right) + \varepsilon (Av, u) + (uv_x, u) + \alpha (vu_x, u) + (\alpha - 4)(uu_x, u) + k(u_x, u) = 0 \\ & \left( \frac{dv}{dt}, u \right) + \varepsilon (Av, u) + (uv_x, u) + \alpha (vu_x, u) + (\alpha - 4)(uu_x, u) + k(u_x, u) \\ & = \frac{1}{2} \frac{d}{dt} (\|u\|^2 + \|u\|^2) + \varepsilon (\|u\|^2 + |Au|^2) - \int_{\Omega} uu_x u_{xx} dx = 0 \end{aligned} \quad (3.22)$$

Similar to the proof to Lemma 1, we know:  $|u|^2 + \|u\|^2 \leq M_1 \triangleq r_8$  (3.23)

From (3.23) we can see that  $|u(x, t)|$  and  $\|u(x, t)\|$  are uniformly bounded. In the other words, the semi-group of the solution operator  $S(t)$  is uniformly bounded in  $L^2(\Omega)$  and  $H^1(\Omega)$

$$\left| \int_{\Omega} uu_x u_{xx} dx \right| \leq \frac{1}{2} \|\nabla u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 \leq \frac{\varepsilon}{2} (\|u\|^2 + |Au|^2) + \frac{k_4^2}{2\varepsilon} (\|u\|^2 + |u|^2)^2$$

Where  $k_3, k_4$  are constants. We get inequalities from (3.22)

$$\frac{1}{2} \frac{d}{dt} (\|u\|^2 + \|u\|^2) + \varepsilon (\|u\|^2 + |Au|^2) \leq \frac{\varepsilon}{2} (\|u\|^2 + |Au|^2) + \frac{k_4^2}{2\varepsilon} (\|u\|^2 + |u|^2)^2 \quad (3.24)$$

Integrating (3.24) over the interval  $[t, t+r]$ , we get:  $\int_t^{t+r} (\|u(x, s)\|^2 + |Au(x, s)|^2) ds \leq \rho$  (3.25)

Let  $B(0, \rho)$  be an open ball in  $L^2(\Omega)$  and  $H^1(\Omega)$ , its radius is  $\rho$ . By simple computing, we know that

$s(t)u_0 \in B(0, \rho)$  for all  $t > t_0$ . We will obtain the uniform estimate of Eq. (2.1)-(2.3) in  $H^2(\Omega)$  as follow.

We take the inner product of (2.1) with  $Au$  in  $\Omega$  to obtain

$$\frac{1}{2} \frac{d}{dt} (\|u\|^2 + |Au|^2) + \varepsilon(|Au|^2 + |\nabla Au|^2) = (2\alpha + 1) \int_{\Omega} uu_x u_{xx} dx + (\alpha - 3) \int_{\Omega} u_x u_{xx}^2 dx - \int_{\Omega} uu_{xx} u_{xxx} dx \tag{3.26}$$

Similar to the proof of Lemma 1, we have

$$\left| \alpha \int_{\Omega} uu_x u_{xx} dx + (-3) \int_{\Omega} u_x u_{xx}^2 dx + \int_{\Omega} uu_{xxx} (-u_{xx}) dx \right| \leq \varepsilon \lambda_1 (|Au|^2 + \|u\|^2) + c_{11} \|u\| |Au| (|Au|^2 + \|u\|^2)$$

where  $c_{10} = \max [c_9 + 2(\alpha - 3)c_8, c_7(2\alpha + 1)]$ ,  $c_{11} = \frac{c_{10}^2}{4\varepsilon\lambda_1}$ .

Based on Young inequality and Poincare inequality, we have

$$\frac{d}{dt} (|Au|^2 + \|u\|^2) \leq 2c_{11} \|u\| |Au| (|Au|^2 + \|u\|^2) \leq c_{11} (|Au|^2 + \|u\|^2)^2.$$

From Gronwall’s inequality we have  $|Au|^2 + \|u\|^2 \leq \frac{\gamma_1}{r} \exp(c_{11}\gamma_1)$ ,  $t > t_0 + r$ , Where  $r, \gamma_1, c_1$  are nonnegative constants. Let  $\rho_1 = (\gamma_1/r) \exp(c_{11}\gamma_1)$ , and then  $|Au|^2 \leq \rho_1$  is the attracting set of  $S(t)$  in  $H^2(\Omega)$ . This completes the proof of Theorem 2.

**Proof of Theorem 3.** We only need to prove that  $S(t)$  is a compact operator, thus we can prove the existence of the global attractor. Take the inner product of (2.2) with  $t^2 \Delta Au$  in  $\Omega$  to obtain

$$\left( \frac{dv}{dt}, t^2 \Delta Au \right) + \varepsilon (Av, t^2 \Delta Au) + (uv_x, t^2 \Delta Au) + 3(vu_x, t^2 \Delta Au) + (\alpha - 4)(wu_x, t^2 \Delta Au) + k(u_x, t^2 \Delta Au) = 0 \tag{3.27}$$

From Agmon’s inequality and Ponincare inequality, we have

$$\frac{1}{2} \frac{d}{dt} (|tAu|^2 + |t\nabla Au|^2) - 2t(|Au|^2 + |\nabla Au|^2) + \varepsilon (|t\nabla Au|^2 + |t\Delta Au|^2) \leq c_{12} \|u\|^{1/2} |Au|^{1/2} (|tAu|^2 + |t\nabla Au|^2),$$

where  $c_{12}$  are constants. From Poincare inequality, we get  $|t\nabla Au|^2 > \lambda_1 |tAu|^2$ ,  $|t\Delta Au|^2 > \lambda_1 |t\nabla Au|^2$ . From the above inequality and (3.27), we get

$$\frac{d}{dt} (|tAu|^2 + |t\nabla Au|^2) + 2\varepsilon (|t\nabla Au|^2 + |t\Delta Au|^2) \leq 2\varepsilon \lambda_1 (|tAu|^2 + |t\nabla Au|^2) + c_{13} \|u\| |Au| (|tAu|^2 + |t\nabla Au|^2) + c_{14} (|Au|^2 + |\nabla Au|^2), \tag{3.28}$$

where  $c_{13} = \frac{c_{12}^2}{2\varepsilon\lambda_1}$ ,  $c_{14} = 16/\varepsilon\lambda_1$ . We have

$$\frac{1}{2} \frac{d}{dt} (|Au|^2 + \|u\|^2) + \varepsilon (|Au|^2 + |\nabla Au|^2) \leq \varepsilon \lambda_1 (|Au|^2 + \|u\|^2) + \frac{c_{11}}{2} (|Au|^2 + \|u\|^2)^2$$

Integrating above inequality over  $[t, t + r]$ , we have

$$\int_t^{t+r} (|Au(x, s)|^2 + |\nabla Au(x, s)|^2) ds \leq (\lambda_1 \rho_1 + \frac{c_{11}}{2\varepsilon} \rho_1^2) r + \frac{\rho_1}{\varepsilon} \triangleq \alpha_2(\lambda_1, \rho_1, \varepsilon). \tag{3.29}$$

We denote (3.28) as follows:

$$\begin{aligned} \frac{d}{dt} (|tAu|^2 + |t\nabla Au|^2) &\leq c_{13} \|u\| |Au| (|tAu|^2 + |t\nabla Au|^2) + c_{14} (|Au|^2 + |\nabla Au|^2) \\ \int_t^{t+r} c_{13} (\|u(x, s)\| |Au(x, s)|) ds &\leq c_{15} \int_t^{t+r} (\|u(x, s)\|^2 + |Au(x, s)|^2) ds \leq c_{15} \rho \triangleq \alpha_3(\rho, \varepsilon) \\ \int_t^{t+r} (|sAu(x, s)|^2 + |s\nabla Au(x, s)|^2) ds &\leq (t+r)^2 \alpha_2 \triangleq \alpha_4 \end{aligned} \tag{3.30}$$

By Gronwall's inequality we have:  $|tAu|^2 + |t\nabla Au|^2 \leq (\frac{\alpha_4}{r} + c_{14}\alpha_2) \exp(\alpha_3) \triangleq E^2(\lambda_1, \rho, \varepsilon, t)$ .

From the above discussion, we finally get  $|\nabla Au| < \frac{E(\lambda_1, \rho, \varepsilon, t)}{t}$ . Because the injection of  $H^3(\Omega)$  into  $H^2(\Omega)$  is compact, we can say that  $S(t)$  is equality continuous. From Ascoli-Arzela's theorem, we know that  $S(t)$  is a compact operator. So  $S(t)$  has the global attractor in  $H^2(\Omega)$ .

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