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On the Weak Solution to the General Shallow Water Wave Equation

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Abstract: In this paper we present some results on the weak solutions to a class of nonlinear dispersive wave equations, named the general shallow water wave equation. We obtain a global weak solution as a limit of viscous approximation under the assumption $u_0 \in H^1(R)$.

Keywords:shallow water wave equation; weak solution; global solution; vanishing viscosity method

1 Introduction

In [1]-[3], Degasperis and Procesi firstly studied the following family of third order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1}$$

where α, c_0, c_1, c_2 and c_3 are real constants and indices denote partial derivatives. When $c_1 = -\frac{\alpha}{2}$, $c_2 = \frac{\varepsilon(\beta-1)}{2}$, $c_3 = \varepsilon$ and replacing c_0 with k, and α^2 with ε in the equation above, we obtain the following equation

$$(u - \varepsilon u_{xx})_t + k u_x + \alpha u u_x + \gamma u_{xxx} = \varepsilon(\beta u_x u_{xx} + u u_{xxx}), \quad x \in R,\ t > 0, \tag{2}$$

where $u(x,t)$ stands for the fluid velocity in the x direction (or equivalently the height of the free surface of water above a float bottom), k is a constant related to the critical shallow water wave speed, and α, β, ε are dispersion parameters. It is necessary to point out that Eq.(1.2) is equivalent to Eq.(1.1) since when $\varepsilon = \alpha^2 = c_3$,$k = c_0$, $\alpha = -2c_1$ and $\beta = 1 + \frac{2c_2}{\varepsilon}$, Eq.(1.2) turns out to be Eq.(1.1). To better understand the common properties of the equation (1.1), we resort to study Eq.(1.2), which is convenient for us to research. We call it *the general shallow water wave equation*.

There are at least three famous equations that satisfy the completely integrability condition within this family: KdV equation (see [4]), Camassa-Holm equation (see [5][16]), and Degasperis-Procesi equation (see [1]-[3]).Besides, Eq.(1.2) includes the b-family of equations (see [7]-[9]) and Fornberg-Whitham equation (see [10]) as special case but they are not completely integrable. The b-family of equations cannot be completely integrable unless $b = 2$ or $b = 3$, i.e., there are only two equations satisfying the completely integrability condition within the peakon b-family of equations: Camassa-Holm equation and Degasperis-Procesi equation.

For $\varepsilon = 0$ in Eq.(1.2), we obtain the KdV equation

$$u_t + k u_x + \alpha u u_x + \gamma u_{xxx} = 0$$

and for $\alpha = 3$, $\beta = 2$, $\gamma = 0$ in Eq.(1.2), we obtain the Camassa-Holm equation

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$$(u - \varepsilon u_{xx})_t + ku_x + 3uu_x = \varepsilon(2u_x u_{xx} + uu_{xxx}).$$

Both of them describe unidirectional shallow water waves. Moreover, all these equations have a bi-Hamiltonian structure, they are completely integrable, they have infinitely many conserved quantities. From a mathematical point of view the Camassa-Holm equation is well studied, see [11] for an extensive list of references. Inparticular, we recall that existence and uniqueness results for global weak solutions have been proved by Coclite et al. [11], Constantin and Escher [12], and Xin and Zhang [13], see also Danchin [14] as well as others.

The remainder of the paper is organized as follow:

In Section 2, we give the precise statements of the main results. Section 3 deals with the viscous approximate solutions and the basic energy estimate on u_ε. The uniform one-sided supernorm estimate and the local space-time higher integrability estimate on $\partial_x u_\varepsilon$ besides the compactness of the viscous approximate solution u_ε are given in Section 4. In Section 5, we examine the asymptotic behavior of the solution $u(t,x)$ to Eq.(1.2).

Notation 1 *We shall use the standard notation $|\cdot|_p$ for the norm of the space $L^p(R)$, $1 \leq p < \infty$, i.e.,$|f|_p = (\int_R |f|^p dx)^{1/p}$. The space $L^\infty = L^\infty(R)$ consists of all essentially bounded, Lebesgue measurable functions with the standard norm*

$$|f|_\infty = |f|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R\backslash e} |f(x)|.$$

And we denote the norm in the Sobolev space $H^s = H^s(R)$ by

$$||f||_s = ||f||_{H^s} = (\int_R (1+|\xi|^2)^s |\hat{f(\xi)}|^2 d\xi)^{1/2}$$

for $s \in R$. Here $\hat{f(\xi)}$ is the Fourier transform of $f(x)$.

2 Main results

Before giving the precise statements of the main results, we introduce the definition of a weak solution to the Cauchy problem (1.2):

Definition 1 *A continuous function $u = u(t,x)$ is said to be a global weak solution to the Cauchy problem (1.2) if*
(1) $u(t,x) \in C([0,\infty) \times R) \cap L^\infty(R_+, H^1(R))$ and

$$\|u\|_{H^1(R)} \leq \|u_0\|_{H^1(R)}, \forall t > 0, \tag{3}$$

(2)$u(t,x)$ satisfies Eq.(1.2) in the sense of distributions and takes on the initial data pointwise.

Theorem 2 *Suppose $u_0 \in H^1(R)$. Then the Cauchy problem (1.2) has an admissible weak solution $u = u(t,x)$ in the sense of Definition 1. Furthermore, the weak solution $u(t,x)$ satisfies the following properties:*
(1) (Oleinik type estimate) There exists a positive constant C depending only on $\|u_0\|_{H^1(R)}$ such that

$$\partial_x u(t,x) \leq \tfrac{1}{t} + C, \forall t > 0, \tag{4}$$

(2) $P(t,x) \in L^\infty(R_+, W^{1,\infty}(R))$ and $\partial_x u(t,x) \in L^p_{loc}(R_+ \times R)$ for any $p < 3$; i.e., for any $0 < T,\ M < +\infty$, there exists a positive constant $C_1 = C_1(T, M, p)$ such that

$$\int_0^T \int_{|x|\leq M} |\partial_x u(t,x)|^p \, dxdt \leq C_1, \forall p < 3, \tag{5}$$

(3) Assume $u(t,x)$ is of one sign, then $u = u(t,x)$ approches zero pointwise as $t \to \infty$; i.e.,

$$\lim_{t \to +\infty} |u(t,x)| = 0, \forall x \in R. \tag{6}$$

3 Viscous approximate solutions

In this section, supposing $\varepsilon = 1$ in Eq.(1.2), we construct the approximate solution sequence $u_\varepsilon = u_\varepsilon(t, x)$ as solutions to the viscous problem (3.1), i.e.,

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \gamma \partial_x u_\varepsilon + \partial_x P_\varepsilon = \nu \partial_x^2 u_\varepsilon, \\ P_\varepsilon = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}[\frac{\alpha-1}{2}u_\varepsilon^2 + \frac{3-\beta}{2}(\partial_x u_\varepsilon)^2 + (k+\gamma)u_\varepsilon](t,y)dy, \\ u_\varepsilon(0,x) = u_{\varepsilon 0}(x). \end{cases} \tag{7}$$

The existence, uniqueness, and basic energy estimate on this approximatesolution sequence are given in the following lemma:

Lemma 3 *Let $\varepsilon > 0$ and $u_{\varepsilon 0} \in H^k(R)$ for some $k \geq 2$. Then there exists a unique solution $u_\varepsilon = u_\varepsilon(t,x) \in C([0,\infty), H^k(R))$ to the Cauchy problem (3.1). Furthermore, u_ε satisfies*

$$\|u_\varepsilon\|_{H^1(R)} \leq \|u_{\varepsilon 0}\|_{H^1(R)}, \forall t > 0, \tag{8}$$

Proof. For completeness, we outline the main idea of the proof, one can check the proof of Lemma 2.1 in [13] for more details. For the convenience of statement, we will omit the subscript ε in $u_\varepsilon(t,x)$ in the following proof.

First, applying Kato's theorem (see [15]) one can obtain the local well-posedness result that for $u_0 \in H^k(R)$ ($k \geq 2$), there exists a positive constant T such that (3.1) has a unique solution $u = u(t,\, x) \in C([0,T); H^k(R)) \cap L^2([0,T); H^{k+1}(R))$.

Second, we prove that if T is the lifespan of the solution $u(t,x)$ and $T < +\infty$, i.e., $u \in C([0,T); H^k(R))$, and

$$\lim_{t\to T} \|u(t,\cdot)\|_{H^k(R)} = +\infty, T < +\infty, \tag{9}$$

then

$$\lim_{t\to T}(\|u(t,\cdot)\|_{L^\infty(R)} + \|\partial_x u(t,\cdot)\|_{L^\infty(R)}) = +\infty. \tag{10}$$

In fact, it follows from the equation in (3.1) that

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|u(t,\cdot)\|_{H^k}^2 + \nu\|\partial_x u(t,\cdot)\|_{H^k}^2 \\ &= \sum_{l=0}^{k}\int_R [-\partial_x^l(u\partial_x u)\partial_x^l u + \gamma(\partial_x^{l+1}u)\partial_x^l u - \partial_x^{l+1}P\partial_x^l u](t,x)dx, \end{aligned} \tag{11}$$

by the Gagliardo-Nirenberg inequality one can conclude that there exists a positive constant C depending only k and ε such that

$$\tfrac{1}{2}\tfrac{d}{dt}\|u(t,\cdot)\|_{H^k}^2 \leq C(\|u(t,\cdot)\|_{L^\infty} + \|\partial_x u(t,\cdot)\|_{L^\infty} + 1)^2\|u(t,\cdot)\|_{H^k}^2.$$

Hence it follows from Gronwall's inequality that (3.3) holds true if and only if (3.4) holds.

Now we show that (3.2) holds on $[0,T)$. In fact, it follows from the equation in (3.1) that

$$\begin{aligned} \partial_t(\partial_x u) + u\partial_x(\partial_x u) &= \nu\partial_x^2(\partial_x u) - \partial_x^2 P - (\partial_x u)^2 + \gamma\partial_x^2 u \\ &= \nu\partial_x^2(\partial_x u) - P + \tfrac{\alpha-1}{2}u^2 + \tfrac{1-\beta}{2}(\partial_x u)^2 + (k+\gamma)u + \gamma\partial_x^2 u, \end{aligned} \tag{12}$$

it follows that

$$\|u(t,\cdot)\|_{H^1}^2 + 2\nu\int_0^t \|\partial_x u(t,\cdot)\|_{H^1}^2 dt = \|u_0\|_{H^1}^2, \forall t \in [0,T)\ , \tag{13}$$

and so

$$\|u\|_{H^1} \leq \|u_0\|_{H^1}, \forall t \in [0, T). \tag{14}$$

Finally, the global existence solution follows from the a priori estimate and the standard continuation argument, and (3.8) holds on $[0, \infty)$, so the proof of the lemma is completed. ∎

4 A priori estimates and precompactness

Let $u_0 \in H^1(R)$ and $u_\varepsilon(t, x)$ be the solution to (3.1) with $u_{\varepsilon 0}(x) = j_\varepsilon * u_0(x)$, where j_ε is the standard Friedrichs mollifier. The long-time existence of $u_\varepsilon(t, x)$ is guaranteed by Lemma 3; furthermore, it satisfies (3.2). To obtain the compactness of this approximate solution sequence $\{u_\varepsilon(t, x)\}$ in $L^2_{loc}(R_+, H^1_{loc}(R))$, a priori estimates in addition to (3.2) are needed.

Proposition 4 *([13]) There exists a positive constant C depending only on $\|u_0\|_{H^1}$ such that*

$$\partial_x u_\varepsilon(t, x) \leq \tfrac{1}{t} + C, \forall t > 0,\ x \in R. \tag{15}$$

Proposition 5 *Let $m = \frac{2k}{2l+1}$ with l, k being positive integers and $l \geq k$. Assume that a, b and T are arbitrarily given finite constants with $a < b$ and $T > 0$. Then there exists a positive constant $C = C(a, b, T, m, \|u_0\|_{H^1})$ independent of ε, such that*

$$\int_0^t \int_a^b |\partial_x u_\varepsilon(t, x)|^{2+m} dx dt \leq C. \tag{16}$$

With the basic energy estimate in Section 3 and the uniform a priori estimates above, we are now ready to obtain the necessary compactness of the viscous approximate solution $u_\varepsilon(t, x)$. We start with the weak compactness in $L^\infty(R_+, H^1(R))$.

Proposition 6 *There exist a subsequence $\{u_{\varepsilon j}(t, x),\ P_{\varepsilon j}(t, x)\}$ of the sequence $\{u_\varepsilon(t, x),\ P_\varepsilon(t, x)\}$ and some functions $\{u(t, x),\ P(t, x)\}$, $u \in L^\infty(R_+, H^1(R))$ and $P \in L^\infty(R_+,\ W^{1,\infty}(R))$, such that*

$$u_{\varepsilon j} \to u \quad as \quad j \to \infty, \tag{17}$$

uniformly on each compact subset of $R_+ \times R$, and

$$P_{\varepsilon j} \to P \quad in \quad L^q_{loc}(R_+ \times R) \tag{18}$$

as $j \to \infty, \forall 1 < q < +\infty$.

Next, we show the stronger convergence result, i.e.,

$$\partial_x u_\varepsilon \to \partial_x u \quad in \quad L^2_{loc}(R_+ \times R) \tag{19}$$

as $\varepsilon \to 0^+$, which will guarantee that $u(t, x)$ is a desired weak solution. First, we state a slight variation of the basic result of the theory of Young measures [16-18].

Lemma 7 *Let $u_{t,x}(\lambda)$ be the Young measure associated with $\{q_\varepsilon(t,x)\} \equiv \{\partial_x u_\varepsilon(t,x)\}$.Then for any continuous function $f = f(\lambda)$ with $f(\lambda) = o(|\lambda|^r)$ and $\partial_\lambda f(\lambda) = o(|\lambda|^{r-1})$ as $|\lambda| \to \infty$ and $r < 2$, and for any $\psi \in L^s_c(R)$ with $\frac{1}{s} + \frac{r}{2} = 1$, we have*

$$\lim_{\varepsilon \to 0^+} \int_R f(q_\varepsilon(t,x))\psi(x)dx = \int_R \overline{f(q)}\psi(x)dx \ , \tag{20}$$

uniformly in each compact subset of R_+, where

$$\overline{f(q)} \triangleq \int_R f(\lambda)d\mu_{t,x}(\lambda) \in C([0,\infty),\ L^{\frac{r'}{r}}_{loc}(R)) \tag{21}$$

with $r' \in (r, 2)$. Moreover, for all $T > 0$, we have

$$\lim_{\varepsilon \to 0^+} \int_0^T \int_R g(q_\varepsilon)\varphi dxdt = \int_0^T \int_R \overline{g(q)}\varphi dxdt, \tag{22}$$

and

$$\lambda \in L^l_{loc}(R_+ \times R \times R,\ dt \otimes dx \otimes d\mu_{t,x}(\lambda)), \tag{23}$$

for all $l < 3$, where $g = g(t,x,\lambda)$ is a continuous function satisfying $g(\lambda) = o(|\lambda|^l)$ as $|\lambda| \to \infty$ for some $l < 3$, and $\varphi = \varphi(t,x) \in L^m([0,T] \times R)$ with $\frac{l}{3} + \frac{1}{m} < 1$.

We furthermore give the following lemma.

Lemma 8 *Let $u_{t,x}(\lambda)$ be the Young measure given in Lemma 7. Then*

$$u_{t,x}(\lambda) = \delta_{\bar{q}(t,x)}(\lambda), \tag{24}$$

for almost all $(t,x) \in R_+ \times R$

Proof. The strategy of the proof of this lemma is quite routine, one can see [13] for the explicit proof and we omit it here. ∎

5 Proof of the theorem

With all the preparations given in the previous section, we are now in a position to prove the main results, i.e., Theorem 2. Let $(u,P)(t,x)$ be the limit of the viscous approximate solutions $(u_\varepsilon, P_\varepsilon)(t,x)$ as $\varepsilon \to 0^+$.It then follows from Lemma 3, Propositions 4 and 6 that $u \in C([0,\infty) \times R) \cap L^\infty(R_+, H^1(R))$, $P \in L^\infty(R_+, W^{1,\infty}(R))$ and (2.1)-(2.2) hold. Taking $\varepsilon \to 0^+$ in (3.1), on can see from Proposition 6 that $(u,P)(t,x)$ will be an admissible weak solution provided that

$$q_\varepsilon \equiv \partial_x u_\varepsilon \to q = \partial_x u \quad in \quad L^2_{loc}(R_+ \times R), \tag{25}$$

as $\varepsilon \to 0^+$. However, (5.1) is now a simple consequence of Lemma 8 and Lemma 7. In fact, it follows (4.6)-(4.7) and Lemma 8 that there exists a subsequence of $\{u_\varepsilon(t,x)\}$, still denoted by itself, such that

$$q_\varepsilon \equiv \partial_x u_\varepsilon \to q = \partial_x u \quad in \quad L^{p_1}_{loc}(R_+, L^{p_2}_{loc}(R)) \quad \forall p_1 < \infty,\ p_2 < 2. \tag{26}$$

This together with Proposition 5 implies

$$q_\varepsilon \equiv \partial_x u_\varepsilon \to q = \partial_x u \quad in \quad L^p_{loc}(R_+ \times R) \quad \forall p < 3 \tag{27}$$

by a simple interpolation, which gives (5.1) immediately.

Furthermore, it follows from (5.3) that

$$\partial_x u \in L^p_{loc}(R_+ \times R), \quad \forall p < 3,$$

hence the local space-time higher integrability estimate, i.e., (2.3) holds.

To finish the proof, it remains to investigate the asymptotic behavior of the solution $u(t,x)$. By our assumption that $u(t,x)$ is of one sign, we will consider the nonnegative solution only. The other case is similar. Let $u(t,x) \geq 0$ be an admissible weak solution to Eq.(1.2) with $\varepsilon = 1$. Then for any $t \in R_+$, integrate Eq.(1.2) over $[0,t] \times (-\infty, x]$ to get

$$\int_{-\infty}^{x} u(t,y)dy + \tfrac{1}{2}\int_0^t u^2(s,x)ds - \gamma \int_0^t u(s,x)ds + \int_0^t P(s,x)ds = \int_{-\infty}^{x} u_0(y)dy. \tag{28}$$

Then we have

$$\|u(t,)\|_{L^1(R)} + \tfrac{1}{2}\|u(,x)\|_{L^2(R_+)} - \gamma\|u(,x)\|_{L^1(R_+)} + \|P(,x)\|_{L^1(R_+)} \leq \|u_0\|_{L^1(R)}. \tag{29}$$

On the other hand, from the standard estimate on convolution one can get

$$\|\partial_x P(t,)\|_{L^2} \leq C_0 \|u_0\|^2_{H^1},$$

which shows that

$$\partial_x u \in L^\infty(R_+, L^2(R)).$$

Hence, there exists a subset $N \in R$ with $meas(N) = 0$ such that $\partial_x u \in L^\infty(R_+)$ for all $x \in R\backslash N$. It follows from this and (5.5) that

$$\lim_{t\to+\infty} |u(t,x)| = 0 \qquad for \qquad x \in R\backslash N. \tag{30}$$

For any $x \in N$, there is a sequence $\{x_j\} \in R\backslash N$, such that $x_j \to x$ as $j \to +\infty$. Since

$$\begin{aligned} |u(t,x)| &\leq |u(t,x_j)| + |u(t,x) - u(t,x_j)| \\ &\leq |u(t,x_j)| + C_0 |x - x_j| \end{aligned},$$

we conclude that

$$\lim_{t\to+\infty} |u(t,x)| = 0 \qquad for \qquad x \in N. \tag{31}$$

Then (2.4) follows from (5.6) and (5.7). This completes the proof of the Theorem 2.

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