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Optimal Control of the Viscous General Shallow Water Wave Equation

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Abstract: This paper studies optimal control problem of the viscous general shallow water wave equation. According to variational method, optimal control theories and distributed system control theories, The optimal control of the viscous general shallow water wave equation under boundary condition is given and the existence of optimal solution is proved.

Keywords: viscous general shallow water wave equation; optimal control; optimal solution

1 Introduction

A. Constantin and D. Lannes proved both the Camassa-Holm and Degasperis-Procesi equations arise in the modeling of the propagation of shallow water waves over a flat bed, which capture stronger nonlinear effects than the classical nonlinear dispersive Benjamin-Bona-Mahoney (also BBM equations) and Korteweg-de Vries equations. They showed that the generalization of the BBM equations:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = 0, \qquad with \quad \alpha - \beta = \frac{1}{6},$$

under the scaling $\mu \ll 1, \varepsilon = O(\mu)$ is provided by the following class of equations:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon\mu(\delta u_x u_{xx} + \gamma u u_{xxx}), \tag{1.1}$$

where ε and μ are two dimensionless parameters defined as : $\varepsilon = a/h$,$\mu = h^2/\lambda^2$.
(here h is the mean depth, a is the typical amplitude and λ the typical wavelength of the waves under consideration) and α, β, are linear dispersive coefficients while δ, γ are nonlinear dispersive coefficients. In fact, the class of equations has been investigated in some papers. In [1]-[3], Degasperis and Procesi firstly studied the following family of third order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.2}$$

where α, c_0, c_1, c_2 and c_3 are real constants and indices denote partial derivatives. When $c_1 = -\alpha/2, c_2 = \varepsilon(\beta - 1)/2, c_3 = \varepsilon$ and replacing c_0 with k, and α^2 with ε in the equation above, we obtain the following equation:

$$\begin{cases} (u - \varepsilon u_{xx})_t + k u_x + \alpha u u_x + \gamma u_{xxx} = \varepsilon(\beta u_x u_{xx} + u u_{xxx}), & x \in R,\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

where $u(x,0)$ is the fluid velocity at time t in the spatial x direction (or equivalently the height of the free surface of water above a float bottom), k is a constant related to the critical shallow water wave speed, and $\alpha, \beta, \varepsilon$ are dispersion parameters. These equations have stronger nonlinearities, which could allow the appearance of wave breaking that is not captured by the BBM equations. Besides, they could account

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correctly for large amplitude waves whose behavior is more nonlinear than dispersive. To better understand the common properties of the equations (1.1) and (1.2), we resort to study Eq.(1.3), which is more convenient for us to research. We call it the general shallow water wave equation. There are at least three famous equations that satisfy the completely integrability condition within this family: KdV equation (see [4]), Camassa-Holm equation (see [5][16]), and Degasperis-Procesi equation (see [1-3]). Up to now, many papers were devoted to above equation's study (see [1-17]). In this paper, we study the optimal control of the viscous general shallow water wave equation $\varepsilon = 1$ in Eq.(1.3) under the periodic boundary condition. The equation is as follows:

$$\begin{cases} u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + ku_x + \gamma u_{xxx} + \alpha uu_x - \beta u_x u_{xx} - uu_{xxx} = f + B^*\varpi \\ u(x,0) = u_0(x) \end{cases}, \qquad (1.4)$$

where $t > 0, x \in \Omega = [0,1]\,, u \in H = L^2(\Omega)$

On the other hand, with the development and application of technology, it is necessary to solve the problems of the optimal control theories with PDE.While the optimal control theories with PDE are much more difficult to deal with. Nowdays,two methods are introduced to study the control theories with PDE:one using low model method, and then changing into ODE model; the other using quasi-optimal control method. No matter which one is chosen, it is necessary to prove the existence of optimal solution according to the basic theories. Now, there are plenty of researches concerned with the optimal control problem. For example, Zhao and Tian studied the optimal control of sufficient nonlinear-Burgers and KdV- Burgers equation [8]. Zhu and Tian studied the optimal control of KdV- Burgers Equation [7]. Zhao studied the optimal control of Kuramto-sivashing equation [10], Tian and Shen studied the optimal control of the b-family equation [14].

The remainder of the paper is organized as follows. In Sec.2, we give some notations, definitions and some Theorem in this paper. Sec.3 is devoted to the study of problem (P). We discuss the optimal control of the viscous general shallow water wave equation and prove the existence of optimal solution.

2 Preliminaries

For fixed $T > 0$, we set $\Omega = (0,1)$ and $Q = (0,T) \times \Omega$. Let $Q_0 \subseteq Q$ be an open set with positive measure.Let $V = H_0^1(0,1)$, and $H = L^2(0,1)$, $V^* = H^{-1}(0,1)$, and $H^* = L^2(0,1)$ are dual spaces respectively. We supply V with the inner product $< \varphi, \psi >_V =< \varphi_x, \psi_x >_H, \forall \varphi, \psi \in V$, Further, the extension operator $B^* \in L(L^2(Q_0), L^2(V^*))$ is given by $B^*q = \begin{cases} q \ in \ Q_0 \\ 0 \ in \ Q \backslash Q_0 \end{cases}$.

Define $\| u \|_{H^m(\Omega)} = \| D^m u \|_H$ where $D^m = \partial^m / \partial x^m, m = 0, 1, 2 \cdots$.

For $T > 0$, the space $L^2(0,T,V)$ and $C(0,T;H)$ denote the space of square integrable and continuous functions, in the sense of Bochner from $[0,T]$ to V The space $W(0,T,V)$ is defined by $W(0,T,V) = \{\varphi : \varphi \in L^2(0,T,V), \varphi_t \in L^2(0,T,V^*)\}$. which is a Hilbert space endowed with common inner product. For brevity we write $L^2(V)$, $C(H)$ and $W(V)$ in the place of $L^2(0,T,V))$, $C(0,T;H)$ and $W(0,T,V)$.

In the paper, we consider the following equation under initial value and boundary condition

$$\begin{cases} u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + ku_x + \gamma u_{xxx} + \alpha uu_x - \beta u_x u_{xx} - uu_{xxx} = f + B^*\varpi \\ u(x,0) = u_0(x) \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \end{cases}, \qquad (2.1)$$

where $x \in (0,1), t \in [0,T], f + B^*\varpi \in L^2(V^*)$ and a control $\varpi \in L^2(Q_0)$.

With $y = u - u_{xx}$, Eq.(2.1) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} y_t - \varepsilon y_{xx} + ku_x + \gamma u_{xxx} + \beta u_x y + uy_x + (\alpha - 1 - \beta)uu_x = f + B^*\varpi \\ y(x,0) = u_0(x) - u_{0,xx}(x) = \phi(x) \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \end{cases}, \qquad (2.2)$$

where $x \in (0,1), t \in [0,t], \phi(x) \in H, f + B^*\varpi \in L^2(V^*)$ and $\varpi \in L^2(Q_0)$.

Definition 2.1 *A function $y(x,t) \in W(V)$ is called a weak solution to Eq.(2.2), if* $\frac{d}{dt}(y,\varphi)_H - \varepsilon(y_{xx},\varphi)_H + (ku_x,\varphi)_H + (\gamma u_{xxx},\varphi)_H + (\beta u_x y,\varphi)_H + (uy_x,\varphi)_H + ((\alpha - 1 - \beta)uu_x,\varphi)_H = \langle f + B^*\varpi, \varphi \rangle_{V^*,V}$ *for $\varphi \in V$, and $t \in [0,T]$, and $y(0) = \phi$ in H are valid.*

Theorem 2.1 *With $\phi(x) \in H$, $f+B^*\varpi \in L^2(V^*)$ holding Eq.(2.2) admits a unique weak solution $y(x,t) \in W(0,T;V)$ in [0,T].*

In order to prove the theorem, we use Galerkin method and a series of mathematical estimates following the step mentioned in ref. ([7-17]).

Lemma 2.1 *If $f \in L^2(V^*)$ and $\phi \in H$, then there exists two constants $C_2, C_2' > 0$*

$$\|y\|^2_{W(V)} \le C_2 \left[(\|\phi\|_H + \|f\|_{2(V^*)})^2 + \|\varpi\|^2_{L^2(Q_0)} \right] + C_2'.$$

Proof. Multiplying Eq.(2.2) by y yields:

$$yy_t - \varepsilon y y_{xx} + k u_x y + \gamma u_{xxx} y + \beta u_x y^2 + u y_x y + (\alpha - \beta - 1) u u_x y = (f + B^*\varpi)\, y$$

Integrating the above equation with respect x to Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|y\|^2_H + \varepsilon \int_0^1 y_x^2 dx - (2\beta - 1)\int_0^1 u y y_x dx - \frac{\beta + 1 - \alpha}{2}\int_0^1 u^2 y_x^2 dx = \langle f + B^*\varpi, y\rangle_{V^*,V}\,, \quad (2.3)$$

Multiplying Eq.(2.2) by u and integrating with respect to x on Ω, we get

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_H + \|u\|^2_V) + \varepsilon(\|u\|^2_V + \|u\|^2_{H^2}) = (\beta - 2)\int_0^1 u u_x u_{xx} dx + \langle f + B^*\varpi, u\rangle_{V^*,V}\,, \quad (2.4)$$

Since $f + B^*\varpi \in L^2(V^*)$ is a control item, we can assume that $\|f + B^*\varpi\|_{V^*} \le M_1$, where M_1 is a positive constant.
Due to basic estimate and soblev embedding theorem, we deduce that

$$\left| \int_0^1 u u_x u_{xx} dx \right| \le \|u_x\|_{L^\infty(\Omega)} \|u_x\|^2_H \le k_1 \|u\|_{H^2(\Omega)} \|u\|^2_V\,, \quad (2.5)$$

where k_1 is a non-negative embedding constant.
It then follows form (2.4), (2.5) and Young's inequality we can get

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}(\|u_m\|^2_H + \|u_m\|^2_V) + \varepsilon(\|u_m\|^2_V + \|u_m\|^2_{H^2}) \\ \le \varepsilon \|u_m\|^2_{H^2} + \frac{(\beta-2)^2 k_1^2}{\varepsilon}\|u_m\|^4_V + \varepsilon\|u_m\|^2_V + \frac{M_1^2}{\varepsilon}, \\ \frac{1}{2}\frac{d}{dt}(\|u_m\|^2_H + \|u_m\|^2_V) \le \frac{(\beta-2)^2 k_1^2}{\varepsilon}\|u_m\|^4_V + \frac{M_1^2}{\varepsilon}, \\ \frac{d}{dt}(\|u_m\|^2_H + \|u_m\|^2_V) \le \frac{2(\beta-2)^2 k_1^2}{\varepsilon}(\|u_m\|^2_H + \|u_m\|^2_V)^2 + \frac{2M_1^2}{\varepsilon}. \end{array} \quad (2.6)$$

It thus transpires that

$$\begin{array}{l} \|u_m\|^2_H + \|u_m\|^2_V \\ \le \frac{M_1}{(\beta-2)K_1}\tan\left\{ \frac{2(\beta-2)K_1 M_1}{\varepsilon} t + \arctan\left[\frac{(\beta-2)K_1(\|u_0\|^2_H + \|u_0\|^2_V)}{M_1} \right] \right\} \triangleq M_2^2, \end{array} \quad (2.7)$$

where $\forall t \in [0, \mathrm{T}]$, and $T \le \pi\varepsilon/4(\beta-2)K_1 M_1$.
From above analysis, we know that $\|u\|_H \le M_2$, $\|u\|_V \le M_2$, where M_2 is a positive constant.
Multiplying Eq.(2.2) by $-u_{xx}$ and use the same argument as before, we have $\|u\|^2_H \le M_3$ where M_3 is a positive constants.
Then by Poincare inequality and Soblev embedding theorem, we can get

$$\begin{array}{l} \left| \int_0^1 u y y_x dx \right| \le \|u\|_{L^\infty(\Omega)} \|y\|_H \|y_x\|_H \le K_2 \|u\|_V (\|y\|^2_H + \|y_x\|^2_H) \le K_2 M_2(\lambda_1 + 1)\|y\|^2_V \\ \left| \int_0^1 u^2 y_x dx \right| \le \|u\|_{L^\infty(\Omega)} \|u\|_H \|y_x\|_H \le K_2 M_2^2 \|y\|_V \end{array}, \quad (2.8)$$

where K_2 is the embedding constant and λ_1 is the Poincare coefficient.
Substituting (2.8) into (2.3), we get

$$\frac{1}{2}\frac{d}{dt}\|y\|^2_H + \varepsilon\int_0^1 y_x^2 dx \le \langle f + B^*\varpi, y\rangle_{V^*,V} + N_1\|y\|^2_V + N_2\|y\|_V\,, \quad (2.9)$$

where $N_1 = (2\beta - 1)K_2M_2(\lambda_1 + 1), N_2 = \frac{\beta+1-\alpha}{2}K_2M_2^2$.
From Holder's inequality, and integrating (2.9) with respect to t on $[0, T]$, we derive that

$$\begin{aligned}&\tfrac{1}{2}\|y(T)\|_H^2 - \tfrac{1}{2}\|\phi\|_H^2 + \varepsilon\|y\|_{L^2(V)}^2 \\ &\le \int_0^T \langle f + B^*\varpi, y\rangle_{V^*,V}\,dt + N_1\|y\|_{L^2(V)}^2 + \tfrac{1}{2\varepsilon}N_2^2 + \tfrac{\varepsilon}{2}\|y\|_{L^2(V)}^2 .\end{aligned} \tag{2.10}$$

From Holder's inequality, we obtain

$$\int_0^T \langle f + B^*\varpi, y\rangle_{V^*,V}dt \le \int_0^T \|f + B^*\varpi\|_{V^*}\|y\|_V dt \le \|f + B^*\varpi\|_{L^2(V^*)}\|y\|_{L^2(V)} . \tag{2.11}$$

Substituting (2.11) into (2.10) and from Young's inequality, we can get

$$\|y(T)\|_H^2 - \|\phi\|_H^2 + (\varepsilon/2 - N_1)\|y\|_{L^2(V)}^2 \le \frac{1}{(\varepsilon/2 - N_1)}\|f + B^*\varpi\|_{L^2(V^*)}^2 + \frac{1}{\varepsilon}N_2^2T. \tag{2.12}$$

It thus transpires that

$$\begin{aligned}&\|y\|_{L^2(V)}^2 \le \tfrac{1}{(\varepsilon/2-N_1)}\|\phi\|_H^2 + \tfrac{1}{(\varepsilon/2-N_1)^2}\|f + B^*\varpi\|_{L^2(V^*)}^2 + \tfrac{1}{\varepsilon(\varepsilon/2-N_1)}N_2^2T \\ &\le \max\left\{\tfrac{1}{(\varepsilon/2-N_1)}, \tfrac{1}{(\varepsilon/2-N_1)^2}\right\}\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 + \tfrac{1}{\varepsilon(\varepsilon/2-N_1)}N_2^2T\ , \\ &\le C_0\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 + C_0'\end{aligned} \tag{2.13}$$

where $C_0 = \max\left\{\frac{1}{(\varepsilon/2-N_1)}, \frac{1}{(\varepsilon/2-N_1)^2}\right\}, C_0' = \frac{1}{\varepsilon(\varepsilon/2-N_1)}N_2^2T$ and $\varepsilon > 2N_1$.
By (2.2) and in view of $\|u\|_H \le M_2, \|u\|_V \le M_2, \|u\|_{H^2} \le M_3$, we thus have

$$\begin{aligned}\|y_t\|_{V^*} &\le \|f + B^*\varpi\|_{V^*} + \varepsilon\|y\|_V + k\|u\|_H + \gamma\|u\|_{H^2} + \beta\|u\|_H\|y\|_V + \|u\|_V\|y\|_H + (\alpha-1-\beta)\|u\|_V\|u\|_H \\ &\le \|f + B^*\varpi\|_{V^*} + \varepsilon\|y\|_V + \beta M_2\|y\|_V + M_2\|y\|_H + \left[kM_2 + \gamma M_3 + (\alpha - 1 - \beta)M_2^2\right] \\ &\le \|f + B^*\varpi\|_{V^*} + (\varepsilon + \beta M_2)\|y\|_V + M_2\|y\|_H + N_3,\end{aligned}$$

where $N_3 = kM_2 + \gamma M_3 + (\alpha - 1 - \beta)M_2^2$. Then

$$\begin{aligned}\|y_t\|_{V^*}^2 &\le 4\|f + B^*\varpi\|_{V^*}^2 + 4(\varepsilon + \beta M_2)^2\|y\|_V^2 + 4M_2^2\|y\|_H^2 + 4N_3^2 \\ &\le 4\|f + B^*\varpi\|_{V^*}^2 + 4\left[(\varepsilon + \beta M_2)^2 + M_2^2\lambda_1\right]\|y\|_V^2 + 4N_3^2,\end{aligned}$$

where λ_1 is the Poincare coefficient. Integrating the above inequality with respect to t on $[0, T]$, we derive that

$$\begin{aligned}&\|y_t\|_{L^2(V^*)}^2 \le 4\|f + B^*\varpi\|_{L^2(V^*)}^2 + 4\left[(\varepsilon + \beta M_2)^2 + M_2^2\lambda_1\right]\|y\|_{L^2(V)}^2 + 4N_3^2T \\ &\le 4\|f + B^*\varpi\|_{L^2(V^*)}^2 + 4\left[(\varepsilon + \beta M_2)^2 + M_2^2\lambda_1\right] \\ &\times\left\{C_0\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 + C_0'\right\} + 4N_3^2T \\ &\le C_1\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 + C_1',\end{aligned} \tag{2.14}$$

where $C_1 = 4 + 4\left[(\varepsilon + \beta M_2)^2 + M_2^2\lambda\right]C_0, C_1' = 4\left[(\varepsilon + \beta M_2)^2 + M_2^2\lambda\right]C_0' + 4N_3^2T$.
Taking into account (2.13) and (2.14) yield

$$\begin{aligned}&\|y\|_{W(V)}^2 = \|y\|_{L^2(V)}^2 + \|y_t\|_{L^2(V^*)}^2 \\ &\le \left\{C_0\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 + C_0'\right\} + \left\{C_1\left(\|\phi\|_H + \|f + B^*\varpi\|_{L^2(V^*)}\right)^2 + C_1'\right\} \\ &\le C_2\left[(\|\phi\|_H^2 + \|f\|_{L^2(V^*)}^2)^2 + \|\varpi\|_{L^2(Q_0)}^2\right] + C_2',\end{aligned}$$

where $C_2 = C_0 + C_1, C_2' = C_0' + C_1'$ ■

3 The optimal control problem of the general shallow water wave equation

In this research we are concerned with distributed control applied to the viscous general shallow water wave equation. As a model we take the distributed optimal control problem

$$(P)\begin{cases} \min J(y,\varpi)=\frac{1}{2}\|Cy-z\|_s^2+\frac{\delta}{2}\|\varpi\|^2_{L^2(Q_0)} \\ y_t-\varepsilon y_{xx}+ku_x+\gamma u_{xxx}+\beta u_x y+uy_x+(\alpha-1-\beta)uu_x=f+B^*\varpi\ in\,(0,T)\times(0,1) \\ y(0)=\phi(x), \qquad \phi(x)\in H \\ u(0,t)=u(1,t)=u_x(0,t)=u_x(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0 \end{cases}$$

where $y=u-u_{xx}$ Let a control $\varpi\in L^2(Q_0)$, y is a weak solution to Eq.(2.2) from Theorem 2.1. Due to $u=(1-\partial_x^2)^{-1}y$ we can infer that there exists a weak solution u to Eq.(2.1). Given an observation operator, in which S is a real Hilbert space and C is continuous. We choose performance index of tracking type

$$J(y,\varpi)=\frac{1}{2}\|Cy-z\|_s^2+\frac{\delta}{2}\|\varpi\|^2_{L^2(Q_0)}, \tag{3.1}$$

where $z\in S$ is a desired state and $\delta>0$ is fixed.
Optimal control problem about Eq.(2.2) is:

$$\min J(y,\varpi), where\,(y,\varpi)\,\text{satisfies Eq}(2.2). \tag{3.2}$$

We set $X=W(V)\times L^2(Q_0)\ and\,Y=L^2(V)\times H$,
define an operator $e=e(e_1,e_2):X\to Y\ by\ e(y,\varpi)=\begin{bmatrix} G \\ y(x,0)-\phi(x)\end{bmatrix}$,
where G $=(-\Delta)^{-1}(y_t-\varepsilon y_{xx}+ku_x+\gamma u_{xxx}+\beta u_x y+uy_x+(\alpha-1-\beta)uu_x-f-B^*\varpi)$,and Δ is an operator from $H_0^1(\Omega)\ to\ H^{-1}(\Omega)$
Then we write (3.2) in the following form

$$\min J(y,\varpi)\ subject\ to\ e(y,\varpi)=0. \tag{3.3}$$

Theorem 3.1 *There exists an optimal control solution to the problem (P).*

Proof. Let $(y,\varpi)\in X$ satisfy the equation $e(y,\varpi)=0$. We have $J(y,\varpi)\geq\frac{\delta}{2}\|\varpi\|_{L^2(Q_0)}$.
From Lemma 2.1, we then deduce that

$$\|y\|_{W(V)}\to\infty\ yields\ \|\varpi\|_{L^2(Q_0)}\to\infty \tag{3.4}$$

Hence,

$$J(y,\varpi)\to\infty\ when\ \|(y,\varpi)\|_x\to\infty \tag{3.5}$$

As the norm is weakly lowered semi-continuous, we achieve that J is weakly lowered semi-continuous.
Since $J(y,\varpi)>0$, all $(y,\varpi)\in X$ hold, and there exist $\zeta>0$ with

$$\zeta=\inf\{J(y,\varpi)\,|(y,\varpi)\in X\ with\ e(y,\varpi)=0\} \tag{3.6}$$

This implies the existence of a minimizing sequence $\{(y_n,\varpi_n)\}_{n\in N}$ in X such that

$$\zeta=\lim_{n\to\infty}J(y^n,\varpi^n)\,and\,e(y^n,\varpi^n)=0\,for\,all\,n\in N$$

Due to (3.5), there exists an element $(y,\varpi)\in X$ with

$$y^n\overset{weak}{\longrightarrow}y^*,n\to\infty,y\in W(V), \tag{3.7}$$

$$\varpi^n\overset{weak}{\longrightarrow}\varpi^*,n\to\infty,\varpi\in L^2(Q_0). \tag{3.8}$$

We can infer from (3.7) that

$$\lim_{n\to\infty}\int_0^T(y_t^n(t)-y_t^*,\varphi(t))_{V^*,V}dt=0,\forall\varphi\in L^2(V). \tag{3.9}$$

Since $W(V)$ is compactly embedded into $L^2(L^\infty)$, we derive that $y^n \to y^*$ strongly in $L^2(L^\infty)$, as $n \to \infty$. Since $W(V)$ is continuously embedded into $C(H)$, we can also derive that $y^n \to y^*$ strongly in $C(H)$, as $n \to \infty$. Then, we can infer that $u^n \to u^*$ strongly in $C(H)$ also.
As the sequence $\{y^n\}_{n\in N}$ converges weakly,$\|y^n\|_{W(V)}$ is bounded. From embedding theorem, we deduce that $\|y^n\|_{L^2(L^\infty)}$ is also bounded. Since $y^n \to y^*$ strongly in $L^2(L^\infty)$, we can infer that $\|y^*\|_{L^2(L^\infty)}$ is bounded. Thus, it follows from Holder's inequality that

$$\left|\int_0^T\int_0^1 (u_x^n y^n - u_x^* y^*)\varphi dxdt\right| \le \left|\int_0^T\int_0^1 u_x^n (y^n - y^*)\varphi dxdt\right| + \left|\int_0^T\int_0^1 (u_x^n - u_x^*) y^*\varphi dxdt\right|$$
$$\le \int_0^T \|y^n - y^*\|_{L^\infty} \|u^n\|_H \|\varphi\|_V \, dt + \int_0^T \|y^*\|_{L^\infty} \|u^n - u^*\|_H \|\varphi\|_V \, dt$$
$$\le \|y^n - y^*\|_{L^2(L^\infty)} \|u^n\|_{C(H)} \|\varphi\|_{L^2(V)} + \|u^n - u^*\|_{C(H)} \|y^*\|_{L^2(L^\infty)} \|\varphi\|_{L^2(V)} \overset{n\to\infty}{\longrightarrow} 0 \;\; for\, \forall\varphi \in L^2(V).$$
$$\left|\int_0^T\int_0^1 (u^n y_x^n - u^* y_x^*)\varphi dxdt\right| \le \left|\int_0^T\int_0^1 (u_x^* y^* - u_x^n y^n)\varphi dxdt\right| + \left|\int_0^T\int_0^1 (u^* y^* - u^n y^n)\varphi_x dxdt\right|$$
$$\le \left|\int_0^T\int_0^1 (u_x^* y^* - u_x^n y^n)\varphi dxdt\right| + \|u^n - u^*\|_{C(H)} \|y^*\|_{L^2(L^\infty)} \|\varphi\|_{L^2(V)}$$
$$+ \|u^n\|_{C(H)} \|y^n - y^*\|_{L^2(L^\infty)} \|u^n\|_{C(H)} \|\varphi\|_{L^2(V)} \overset{n\to\infty}{\longrightarrow} 0 \;\; \text{for } \forall\varphi \in L^2(V)$$

From (3.8), we get $\left|\int_0^T\int_0^1 (B^*\varpi^n - B^*\varpi^*)\varphi dxdT\right| \overset{n\to\infty}{\longrightarrow} 0$ for $\forall\varphi \in L^2(V)$.
In view of the above discussion, we can conclude that $e_1(y^*, \varpi^*) = 0, \forall n \in N$.
From $y^* \in W(V)$, we derive that $y^*(0) \in H$.
Since $y^n \overset{weak}{\longrightarrow} y^*$ in $W(V)$, we can infer that $y^n \overset{weak}{\longrightarrow} (0)y^*(0)$, when $n \to \infty$.
Thus we obtain $\langle y^n(0) - y^*(0), \psi\rangle_H \overset{weak}{\longrightarrow} 0, \forall\psi \in H$.
Consequently, we can derive that $e(y^*, \varpi^*) = 0 \; in \; Y$.

In conclusion, there exists an optimal solution (y^*, ϖ^*) to the problem (P). In the meantime, we can infer that there exists an optimal solution (u^*, ϖ^*) to the viscous general shallow water wave equation due to $y = u - u_{xx}$. ■

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